Justification of lubrication approximation: an application to fluid/solid interactions

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Abstract

We consider the stationary Stokes problem in a three-dimensional fluid domain $\mathcal{F}$ with non-homogeneous Dirichlet boundary conditions. We assume that this fluid domain is the complement of a bounded obstacle $\mathcal{B}$ in a bounded or an exterior smooth container Ω. We compute sharp asymptotics of the solution to the Stokes problem when the distance between the obstacle and the container boundary is small.

Keywords. Fluid/solid interactions, Stokes problem, lubrication approximation.

In this paper, we consider the 3D-Stokes problem

$$\Delta \mathbf{u} - \nabla p = 0\,, \tag{1}$$
$$\nabla \cdot \mathbf{u} = 0\,, \tag{2}$$

in a fluid domain $\mathcal{F}$. Without restricting the generality, we set the viscosity of the fluid to 1. We assume that $\mathcal{F} = \Omega \setminus \overline{\mathcal{B}}$ is the complement of a smooth simply-connected bounded domain $\mathcal{B}$ inside a container Ω. The container Ω is either a relatively compact simply-connected smooth open set or the exterior of a simply-connected smooth compact set $\mathcal{B}^*$. In both cases, it has a smooth compact connected boundary. We complete then (1)-(2) with boundary conditions:

$$\mathbf{u} = \mathbf{u}^*\,, \quad \text{on } \partial\Omega\,, \tag{3}$$
$$\mathbf{u} = \mathbf{0}\,, \quad \text{on } \partial\mathcal{B}\,, \tag{4}$$

where $\mathbf{u}^* \in C^\infty(\partial\Omega)$ does not prescribe any flux through the container boundary:

$$\int_{\partial\Omega} \mathbf{u}^* \cdot \mathbf{n} \mathrm{d}\sigma = 0\,. \tag{5}$$

If Ω is an exterior domain, we add a vanishing condition at infinity:

$$\lim_{|(x,y,z)|\to\infty} \mathbf{u}(x,y,z) = 0\,. \tag{6}$$

With the above assumptions on the boundary data $\mathbf{u}^*$ it is classical that system (1)–(4)(+(6)) admits a unique classical solution $(\mathbf{u}, p)$ (the pressure p being unique up to a constant), see [8] for instance. *Our aim in this paper is to give a sharp description of the solution to the Stokes problem* (1)–(4)(+(6)) *when the distance between* $\mathcal{B}$ *and* $\partial\Omega$ *is small.*

Computing such asymptotics is an important issue related to the modeling of solid-body motion inside a viscous fluid. The typical configuration we have in mind is that $\mathcal{B}$ (resp. $\mathcal{B}$ and $\mathcal{B}^*$) is a

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(are) moving solid body (bodies) inside the container Ω (in the whole space) which is filled by a viscous incompressible constant-density fluid. A typical issue is then to determine whether the fluid viscosity prevents the moving body $\mathcal{B}$ from touching other solid boundaries (*i.e.* the boundary of the container or the boundary of the other solid body) and more generally to measure the influence of the viscosity on the close-contact dynamics of $\mathcal{B}$. To this end, one remarks that, when a solid body is about to collide another solid boundary with moderate relative velocity, the fluid Reynolds number tends to 0 in the gap so that a stationary Stokes system is sufficient to predict the force and torque exherted by the fluid on the moving body. With this particular application in mind, several authors consider the free-fall of a sphere above a ramp [4, 5, 7, 18, 19] in a Stokes fluid. Explicit values for the solution to the Stokes problem (1)–(4)(+(6)) and the associated force and torque are provided. A formal lubrication approximation is also proposed in [6, 11] which generalizes these formulas to arbitrary configurations. In the limit regime where there is contact, solutions to the Stokes problem are also computed in [17] under further assumptions on the boundary data $\mathbf{u}^*$ (broadly, the boundary data $\mathbf{u}^*$ has to vanish sufficiently where there is contact). Related computations for a perfect fluid are provided in [1, 15, 16].

In this paper, we fill the gap between the explicit formulas of [4, 5, 7, 18, 19], the formal asymptotics of [6, 11] and the analysis in [17]. We justify rigorously lubrication approximation in the spirit of [2, 3]. Compared to these latter references, we are interested inhere in fluid films that do not vanish uniformly in their widths. This fact leads to severe new difficulties. First, the lubrication scaling acts on coordinates in both tangential and orthogonal directions to the boundaries (see (9)–(12)). Second, the asymptotic pressure and velocity-fields yielding from the formal lubrication approximation are defined on an asymptotic fluid domain which is not simply related to the fluid-domain for a given positive body/boundary distance h. Consequently, in order to compare the values of the solution to the Stokes problem with its asymptotic value, we introduce an intermediate velocity-field which embeds the lubrication approximation in the effective fluid-domain (see Section 3.1). The construction of this intermediate velocity-field is an important step of our analysis. Indeed, the intermediate velocity-field is a key-ingredient in order to extend the computations on the close-contact dynamics of bodies in a Stokes fluid to more complicated models: Navier-Stokes/Newton models or Navier-Stokes/elasticity models. This fact has already been shown for the Navier-Stokes/Newton model in simple configurations (see [9, 12, 13, 14]). We believe our approach extends to other incompresible models (for instance to the case of potential flows as in [16]). However, the incompressibility condition is crucial to our computation and the extension to the compressible case is completly open.

To fix ideas, we make more specific the geometry of the gap between $\mathcal{B}$ and $\partial\Omega$. We introduce a set of cartesian coordinates $(x,y,z)\in\mathbb{R}^3$ and $(r,\theta,z)\in(0,\infty)\times(-\pi,\pi)\times\mathbb{R}$ the corresponding cylindrical coordinates. These coordinates are associated with two orthonormal basis of $\mathbb{R}^3$ denoted by $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ and $(\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z)$ respectively. We consider that, in a neighborhood $\mathcal{U}$ of the origin, Ω and $\mathcal{B}$ satisfy:

$$(x,y,z)\in(\mathbb{R}^3\setminus\overline{\Omega})\cap\mathcal{U} \quad\Leftrightarrow\quad \{(x,y)\in B(\mathbf{0},L) \text{ and } z<\gamma_b(x,y)\}\,, \tag{7}$$

$$(x,y,z)\in\mathcal{B}\cap\mathcal{U} \quad\Leftrightarrow\quad \{(x,y)\in B(\mathbf{0},L) \text{ and } z>h+\gamma_t(x,y)\}\,. \tag{8}$$

Here (h,L) are given positive parameters. The first one measures the distance between $\partial\Omega$ and $\partial\mathcal{B}$, the second one is a characteristic length on which parametrizing the boundaries of Ω and $\mathcal{B}$ by (x,y)-variables is relevant. The two functions γ_t and γ_b are smooth on $B(\mathbf{0},L)\subset\mathbb{R}^2$ and we assume throughout the paper that they satisfy the following assumptions:

(A1) $\nabla\gamma_t(\mathbf{0})=\nabla\gamma_b(\mathbf{0})=\mathbf{0}$,

(A2) there exists $R_1>0$ such that (in the sense of symmetric matrices)

$$\nabla^2\gamma_t(x,y)-\nabla^2\gamma_b(x,y)\geq\frac{1}{R_1}\mathbb{I}_3\,,\quad\forall\,(x,y)\in B(\mathbf{0},L)\,.$$

We complement these local assumptions in $\mathcal{U}$ with a global one concerning $\mathcal{B}$ and $\partial\Omega$ outside $\mathcal{U}$:

$$\exists\, \delta > 0 \text{ s.t. } \ \mathrm{dist}(\partial\Omega \setminus \mathcal{U}, \mathcal{B}) > \delta. \tag{A3}$$

The parametrizations (7) and (8) together with assumptions (A1)-(A2)-(A3) are paradigmatic of what we call a "single non-degenerate contact". Indeed, considering that $\mathcal{B}$ is a translating particle inside the container Ω amounts to let the parameter h depend on time. Assumption (A3) implies then that a contact between $\mathcal{B}$ and $\partial\Omega$ may only hold in $\mathcal{U}$. Assumption (A2) yields that the contact in $\mathcal{U}$ is unique and non-degenerate *i.e.* when $h = 0$ the "vertical" distance $\gamma_t - \gamma_b$ vanishes in 0 only and with minimal vanishing order. We emphasize that, in the smooth case we consider here, the uniform boundedness we require in (A2) reduces to assuming that

$$\nabla^2\gamma_t(\mathbf{0}) - \nabla^2\gamma_b(\mathbf{0}) \geq \frac{1}{R_1}\mathbb{I}_3\,, \tag{A'2}$$

up to restrict the size of L and change the value of R_1. In this "single non-degenerate contact"-case, the assumptions (7)-(8) together with (A1) do not restrict the generality: they only amount to choose the origin of coordinates in the only point of $\partial\Omega$ realizing the distance between $\partial\Omega$ and $\partial\mathcal{B}$, and to choose the system of coordinates so that the common normal to $\partial\Omega$ and $\partial\mathcal{B}$, in the pair of points realizing the distance between $\partial\Omega$ and $\partial\mathcal{B}$, is $\mathbf{e}_z$.

Our aim is to compute the asymptotics of the solution to the Stokes problem (1)–(4)(+(6)) for a given boundary condition $\mathbf{u}^* \in C^\infty(\partial\Omega)$ in the geometry depicted above under the further assumption that h is small and other parameters are of order 1. To introduce our main result, we recall the main steps of the formal computations in [6] for the case where $\mathcal{B}$ is a sphere of radius S and $\Omega = \mathbb{R}^3 \setminus \mathcal{B}^*$ with $\mathcal{B}^*$ a sphere of radius R. First, given the shape of the aperture between both spheres, one looks for a solution $(\mathbf{u}, p) := ((u_x, u_y, u_z), p)$ that reads

$$u_x(x,y,z) \;=\; h^{\alpha-\frac{1}{2}}\tilde{u}_x(h^{-\frac{1}{2}}x, h^{-\frac{1}{2}}y, h^{-1}z)\,, \tag{9}$$

$$u_y(x,y,z) \;=\; h^{\alpha-\frac{1}{2}}\tilde{u}_y(h^{-\frac{1}{2}}x, h^{-\frac{1}{2}}y, h^{-1}z)\,, \tag{10}$$

$$u_z(x,y,z) \;=\; h^{\alpha}\tilde{u}_z(h^{-\frac{1}{2}}x, h^{-\frac{1}{2}}y, h^{-1}z)\,, \tag{11}$$

$$p(x,y,z) \;=\; h^{\alpha-2}\tilde{p}(h^{-\frac{1}{2}}x, h^{-\frac{1}{2}}y, h^{-1}z)\,, \tag{12}$$

in the aperture between the spheres. The parameter α is chosen depending on the values of $\mathbf{u}^* := (u_x^*, u_y^*, u_z^*)$. For instance, if $u^* = u_\perp^*\mathbf{e}_z$ with $u_\perp^* \in \mathbb{R}\setminus\{0\}$, one chooses $\alpha = 0$. We proceed with this particular case. We denote with tildas the new space variables

$$\tilde{\mathbf{x}} := (h^{-\frac{1}{2}}x, h^{-\frac{1}{2}}y, h^{-1}z)\,.$$

These new coordinates belong to the set $\tilde{\mathcal{G}}_h$ that "converges" (when $h \to 0$) to

$$\tilde{\mathcal{G}}^{lub} := \left\{(\tilde{x},\tilde{y},\tilde{z}) \in \mathbb{R}^3 \ \text{ s.t. } \ \tilde{z} \in \left(-\frac{\tilde{x}^2+\tilde{y}^2}{2R}, 1 + \frac{\tilde{x}^2+\tilde{y}^2}{2S}\right)\right\}.$$

Substituting ansatz (9)–(12) into (1)–(2) yields:

$$\begin{aligned} \partial_{\tilde{x}}\tilde{u}_x + \partial_{\tilde{y}}\tilde{u}_y + \partial_{\tilde{z}}\tilde{u}_z &= 0\,,\\ \partial_{\tilde{z}\tilde{z}}\tilde{u}_x - \partial_{\tilde{x}}\tilde{p} &= 0\,,\\ \partial_{\tilde{z}\tilde{z}}\tilde{u}_y - \partial_{\tilde{y}}\tilde{p} &= 0\,,\\ \partial_{\tilde{z}}\tilde{p} &= 0\,, \end{aligned}$$

completed with boundary conditions:

$$\begin{aligned} \tilde{u}_x = \tilde{u}_y = 0 \qquad & \text{on } \partial\tilde{\mathcal{G}}^{lub}\,,\\ \tilde{u}_z = 0 \qquad & \text{on } \partial\tilde{\mathcal{G}}_2^{lub} := \left\{\tilde{z} = \tilde{\gamma}_2(\tilde{x},\tilde{y}) =: 1 + \frac{\tilde{x}^2+\tilde{y}^2}{2S}\,, (\tilde{x},\tilde{y}) \in \mathbb{R}^2\right\},\\ \tilde{u}_z = u_\perp^* \qquad & \text{on } \partial\tilde{\mathcal{G}}_1^{lub} := \left\{\tilde{z} = \tilde{\gamma}_1(\tilde{x},\tilde{y}) =: -\frac{\tilde{x}^2+\tilde{y}^2}{2R}\,, (\tilde{x},\tilde{y}) \in \mathbb{R}^2\right\}. \end{aligned}$$

The pressure is normalized by assuming that it vanishes at infinity. Introducing $\tilde{\gamma} := \tilde{\gamma}_2 - \tilde{\gamma}_1$, the unique solution to this problem reads $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = \mathbf{u}^{lub}(\tilde{\mathbf{x}})$, $\tilde{p}(\tilde{\mathbf{x}}) = p^{lub}(\tilde{x}, \tilde{y})$ with:

$$\begin{aligned}
u_x^{lub}(\tilde{x}, \tilde{y}, \tilde{z}) &= \frac{1}{2}\partial_{\tilde{x}} p^{lub}(\tilde{x}, \tilde{y})(\tilde{z} - \tilde{\gamma}_1(\tilde{x}, \tilde{y}))(\tilde{z} - \tilde{\gamma}_2(\tilde{x}, \tilde{y})) \\
u_y^{lub}(\tilde{x}, \tilde{y}, \tilde{z}) &= \frac{1}{2}\partial_{\tilde{y}} p^{lub}(\tilde{x}, \tilde{y})(\tilde{z} - \tilde{\gamma}_1(\tilde{x}, \tilde{y}))(\tilde{z} - \tilde{\gamma}_2(\tilde{x}, \tilde{y})) \\
u_z^{lub}(\tilde{x}, \tilde{y}, \tilde{z}) &= \frac{1}{2}\mathrm{div}_{\tilde{x},\tilde{y}} \left[\int_{\tilde{z}}^{\tilde{\gamma}_2(\tilde{x},\tilde{y})} \left((s - \tilde{\gamma}_2(\tilde{x}, \tilde{y}))(s - \tilde{\gamma}_1(\tilde{x}, \tilde{y})) \right) \nabla_{\tilde{x},\tilde{y}} p^{lub}(\tilde{x}, \tilde{y}) \mathrm{d}s \right],
\end{aligned}$$

and where p^{lub} is the unique solution to

$$-\frac{1}{12}\mathrm{div}(\tilde{\gamma}^3 \nabla p^{lub}) = u_\perp^*, \quad \text{on } \mathbb{R}^2, \tag{13}$$

$$\lim_{|(\tilde{x},\tilde{y})| \to \infty} p^{lub}(\tilde{x}, \tilde{y}) = 0. \tag{14}$$

Herein, we justify these formal computations rigorously. A (nonetheless formal) statement of our main result reads:

Theorem 1. *Assume* (A1)–(A3) *are in force and* $\mathbf{u}^* \in C^\infty(\partial\Omega)$ *satisfies* (5). *Let* $\mathbf{v}_{/\!/}^*$ *and* $v_\perp^*$ *be given by:*

$$\mathbf{v}_{/\!/}^* = u_x^*(\mathbf{0})\mathbf{e}_x + u_y^*(\mathbf{0})\mathbf{e}_y, \quad v_\perp^*(x, y) = u_z^*(\mathbf{0}) + x\partial_x u_z^*(\mathbf{0}) + y\partial_y u_z^*(\mathbf{0}). \tag{15}$$

When $h << 1$ *while* $(\gamma_t, \gamma_b, R_1, \mathbf{u}^*)$ *remain of order* 1, *the main contribution to the velocity-field of the solution* $(\mathbf{u}, p)$ *to the Stokes problem* (1)–(4)(+(6)) *is given by* $\mathbf{v} = (\mathbf{v}_{/\!/}, v_\perp)$ *with*

$$\mathbf{v}_{/\!/}(x, y, z) = \frac{1}{2}(z - (h + \gamma_t(x, y)))(z - \gamma_b(x, y))\nabla_{x,y} q(x, y) + \frac{(h + \gamma_t(x, y) - z)}{\gamma(x, y)}\mathbf{v}_{/\!/}^* \tag{16}$$

$$\begin{aligned}
v_\perp(x, y, z) &= \frac{1}{2}\mathrm{div}_{x,y} \left[\int_z^{h+\gamma_t(x,y)} \left((s - \gamma_b(x, y))(s - (h + \gamma_t(x, y))) \right) \nabla_{x,y} q(x, y) \mathrm{d}s \right] \qquad (17) \\
&\quad + \int_z^{h+\gamma_t(x,y)} \mathrm{div}_{x,y} \left[\frac{(h + \gamma_t(x, y) - s)}{\gamma(x, y)} \mathbf{v}_{/\!/}^* \right] \mathrm{d}s
\end{aligned}$$

in the aperture domain $\mathcal{G}_{L/2} := \{(x, y, z) \in \mathbb{R}^3$ *s.t.* $(x, y) \in B(\mathbf{0}, L/2)$ *and* $z \in (\gamma_b(x, y), h + \gamma_t(x, y))\}$, *and where* q *is the unique solution to*

$$-\frac{1}{12}\mathrm{div}(\gamma^3 \nabla q) = v_\perp^* - \frac{1}{2}\mathrm{div}_{x,y}((\gamma_t + \gamma_b)\mathbf{v}_{/\!/}^*), \quad \text{on } B(\mathbf{0}, L), \tag{18}$$

$$q = 0 \qquad \text{on } \partial B(\mathbf{0}, L), \tag{19}$$

with $\gamma := h + \gamma_t - \gamma_b$.

We introduce here the notations $_{/\!/}$ and $\perp$ for the components of a vector $\mathbf{v}$ that are respectively parallel and normal to the tangent space to $\partial\Omega$ in the origin. Corresponding decomposition of ∇ are denoted by $\nabla_{x,y}$ and ∂_z. We keep this convention in what follows. Our geometric assumptions imply that $\mathbf{e}_x$ and $\mathbf{e}_y$ are tangent to $\partial\Omega$ in the origin. Hence, the derivatives $\partial_x \mathbf{u}^*(\mathbf{0})$ and $\partial_y \mathbf{u}^*(\mathbf{0})$ are well-defined. Even though the result concerns an asymptotic behavior when $h \to 0$ the value of this parameter is fixed in all computations. For this reason, we do not let the parameter h appear in most of our notations (such as the fluid domain $\mathcal{F}$, the distance function γ ...).

A more quantitative statement of the above theorem is given in next section. In particular, we make precise for which norms the extracted contribution is the dominating term, the size of remainder terms and the dependencies w.r.t. $\gamma_t, \gamma_b, R_1, \mathbf{u}^*$. We prove this result by

- computing *a priori* estimates on the pressure q as constructed in this statement (in the regime $h << 1$)

- relating the difference between the exact solution $\mathbf{u}$ to the Stokes problem and (an extension of) $\mathbf{v}$ to the obtained estimates on the pressure q.

As a corollary, we obtain that the leading order in the asymptotics of the Stokes solution is given by the first order expansion of the boundary data $\mathbf{u}^*$ in the origin. This dependance occurs through a pressure solution to a simpler problem but still not explicit. We explain in next section that in some special case (when $\mathcal{B}$ is a sphere and $\Omega = \mathbb{R}^3 \setminus \mathcal{B}^*$ with $\mathcal{B}^*$ a sphere) we can compute accurate informations on the pressure and get explicit expansion of the Stokes solution w.r.t. h and the boundary data $\mathbf{u}^*_{/\!/}(\mathbf{0}), u^*_{\perp}(\mathbf{0}), \nabla_{x,y} u^*_{\perp}(\mathbf{0})$.

In the set of assumptions we introduced up to now, we enforced the boundary condition to vanish on $\partial\mathcal{B}$. We can always reduce asymptotic computations with general Dirichlet boundary conditions to this case. We also emphasize that the computations, that we present here in the "single non-denegerate contact" case, extend to general "non-degenerate contacts". A general non-degenerate contact would correspond to the case where the moving solid $\mathcal{B}$ may collide $\partial\Omega$ in several points satisfying assumption (A2). There would then exist at most a finite number of "contact points" that might be treated separately as "non-degenerate isolated contacts". The linearity of the Stokes problem ensures that the exact solution behaves as the sum of the asymptotic solutions that we compute in the vicinity of each contact point. Previous calculations due to V. Starovoitov show that these "non-degenerate isolated contacts" are the only remaining ones to rule out in order to show that no contact between solid bodies occur in a Stokes or a Navier-Stokes fluid (see [20]).

The outline of the paper is as follows. In next section, we recall the classical theory for solving the Stokes problem (1)–(4)(+(6)) and give a quantitative statement of our main results, see Theorem 4 and Theorem 5. In Section 2, we study the properties of the problem (18)-(19). We compute estimates satisfied by the solution q with respect to the data $(v^*_{\perp}, \mathbf{v}^*_{/\!/})$ and compute the divergence-rate of q when $h \to 0$. One particular feature of the estimates we obtain is that they are "uniform" w.r.t. the distance h. The last two sections are devoted to the proofs of Theorem 4 and Theorem 5 respectively.

1 Quantitative statement of main results

In this section, we recall function-spaces and existence results for problem (1)–(4)(+(6)). We give then a quantitative statement of our main result. We conclude by exhibiting two criterions which measure whether a velocity-field is a good approximation of a solution to the Stokes problem or not.

1.1 Solving the Stokes problem

We first recall the way the system (1)–(4)(+(6)) is tackled in [8]. Let denote:

- $\mathcal{V} := \{\mathbf{u}_{|\mathcal{F}}, \mathbf{u} \in C^\infty_c(\mathbb{R}^3) \text{ s.t. } \nabla \cdot \mathbf{u} = 0\}$ and $\mathcal{V}_0 := \{\mathbf{u} \in C^\infty_c(\mathcal{F}) \text{ s.t. } \nabla \cdot \mathbf{u} = 0\}$,
- V (resp. V_0) the completion of $\mathcal{V}$ (resp. $\mathcal{V}_0$) endowed with the norm:

$$\|\mathbf{u}; V\| := \left[\int_{\mathcal{F}} |\nabla \mathbf{u}|^2\right]^{\frac{1}{2}}.$$

This makes $(V, \|\cdot\,; V\|)$ (resp. $(V_0, \|\cdot\,; V\|)$) to be a Hilbert space endowed with the scalar product:

$$((\mathbf{u}, \mathbf{v})) = \int_{\mathcal{F}} \nabla \mathbf{u} : \nabla \mathbf{v}\,.$$

In the terminology of [8], our space V (resp. V_0) is a closed subspace of $D^{1,2}(\mathcal{F})$ (resp. $D^{1,2}_0(\mathcal{F})$), see [8, p.80]. In particular, applying [8, Theorem II.6.1 (i), (II.6.22)] we have that V embeds in

$L^6(\mathcal{F})$ so that $V \subset W^{1,2}_{loc}(\mathcal{F})$. This entails that $\mathbf{v} \in V$ vanishes at infinity in a weak sense, if required, and has a well-defined trace $\mathbf{v}_{|\partial\mathcal{F}} \in H^{\frac{1}{2}}(\partial\mathcal{F})$. In particular, for arbitrary $\mathbf{u}^* \in H^{\frac{1}{2}}(\partial\Omega)$, we might define the affine-subspace of V:

$$V[\mathbf{u}^*] := \{\mathbf{u} \in V \text{ such that } \mathbf{u}_{|\partial\mathcal{B}} = 0 \text{ and } \mathbf{u}_{|\partial\mathcal{B}_*} = \mathbf{u}^*\}\,.$$

We recall that this set is not empty as soon as $\mathbf{u}^*$ prescribes no flux through $\partial\Omega$ and that we have the identity $V[0] = V_0$ (see [8, Theorem II.7.1 (i)]).

Following [8] we introduce the definition of generalized solution to (1)–(4)(+(6)):

Definition 2. *Let $\mathbf{u}^* \in H^{\frac{1}{2}}(\partial\Omega)$ we call generalized solution to* (1)–(4)(+(6)) *any $\mathbf{u} \in V[\mathbf{u}^*]$ such that*

$$\int_{\mathcal{F}} \nabla\mathbf{u} : \nabla\mathbf{v} = 0\,, \quad \forall\, \mathbf{v} \in V_0\,. \tag{20}$$

and we have the classical theorem (see [8, Theorem V.2.1 and Theorem V.1.1]):

Theorem 3. *Given $\mathbf{u}^* \in H^{\frac{1}{2}}(\partial\Omega)$, such that :*

$$\int_{\partial\Omega} \mathbf{u}^* \cdot \mathbf{n}\,\mathrm{d}\sigma = 0\,, \tag{21}$$

there exists a unique generalized solution $\mathbf{u}$ to (1)–(4)(+(6)). *Moreover,*

- *there exists $p \in L^2_{loc}(\mathcal{F})$ such that* (1) *holds in the sense of distributions,*
- *if $\mathbf{u}^* \in C^\infty(\partial\Omega)$ there holds $(\mathbf{u}, p) \in C^\infty(\overline{\mathcal{F}}; \mathbb{R}^4)$.*

A consequence of this result is that generalized solutions with smooth data are classical solutions. As we only consider this particular case throughout the paper, we drop the adjective "generalized" in what follows. Also, we abusively call solution to (1)–(4)(+(6)) a velocity-field $\mathbf{u}$, even though a solution to the Stokes problem is a pair velocity/pressure $(\mathbf{u}, p)$.

1.2 Main results

The main contribution of this paper is the following theorem:

Theorem 4. *Assume* (A1)–(A3) *and boundary condition $\mathbf{u}^* \in C^\infty(\partial\Omega)$ satisfies* (21). *Let denote by $\mathbf{u} \in V$ the unique associated solution to* (1)–(4)(+(6)). *If $h < 1$, there exists $\mathbf{v} \in V$ and a constant $K := K(R_1, C_b, \delta, \partial\Omega)$ such that:*

- *$\mathbf{v}$ is given by* (16)-(17) *in $\mathcal{G}_{L/2}$*
- *If $\|\gamma_t; C^3(B(\mathbf{0}, L))\| + \|\gamma_b; C^3(B(\mathbf{0}, L))\| \le C_b$ and $\|\mathbf{u}^*; H^3(B(\mathbf{0}, L))\| + \|\mathbf{u}^*; H^{\frac{1}{2}}(\partial\Omega)\| \le C_b$ there holds:*
$$\|\mathbf{u} - \mathbf{v}; V\| \le K\left(1 + |u^*_z(\mathbf{0})||\ln(h)|^{\frac{1}{2}}\right).$$

If the distance function γ is moreover radial, we have the following improvement of the second assertion:

$$\|\mathbf{u} - \mathbf{v}; V\| \le K.$$

We remark that $\mathbf{u}$ and $\mathbf{v}$ are bounded uniformly in h as along as this parameter ranges a compact subset of $(0, 1]$. Consequently, this theorem brings relevant informations in the limit $h \to 0$. It shows in particular that even though the fluid is incompressible (so that the Stokes problem is non-local), when h goes to 0 the leading order of the velocity-field is completely fixed by the boundary conditions and the geometric properties of the boundaries in the aperture $\mathcal{G}_L$. It will be clear from the example given below that the velocity-field $\mathbf{v}$ we exhibit is larger than the remainder in the sense of the V-norm. We mention that our first interest in this problem was to compute the diverging term in the case of spheres. As the remainder is bounded in this radial case, we did not look for a more complicated expansion of the solution. Nevertheless, a corollary of our method is the construction of the linear problem on which could rely the computation of a full expansion of the solution in terms of h. However, we would only describe the solution in the gap $\mathcal{G}_L$ and the computation of $O(1)$-terms would require another tool enabling to compute the expansion of the solution outside the gap also.

To highlight the relevance of the above theorem, we detail the case where $\mathcal{B}$ is a sphere and $\Omega = \mathbb{R}^3 \setminus \mathcal{B}^*$ with $\mathcal{B}^*$ a sphere, as in the computations in the introduction. Keeping the convention that $\mathcal{B}^*$ has radius R and $\mathcal{B}$ has radius S, we have that, close to the origin, the functions γ_t and γ_b satisfy:

$$\begin{aligned}
\gamma_t(x,y) &= \frac{x^2+y^2}{2S} + O((x^2+y^2)^{\frac{3}{2}}) \,,\\
\gamma_b(x,y) &= -\frac{x^2+y^2}{2R} + O((x^2+y^2)^{\frac{3}{2}}) \,,
\end{aligned}$$

and also that γ satisfies:

$$\gamma(x,y) = \frac{x^2+y^2}{2R_1} + \frac{(x^2+y^2)^2}{8R_3^3} + O((x^2+y^2)^3)$$

with $(R_1, R_3) \in (0,\infty)^2$ given by:

$$\frac{1}{R_1} = \frac{1}{S} + \frac{1}{R}\,, \quad \frac{1}{R_3^3} = \frac{1}{S^3} + \frac{1}{R^3}\,. \tag{22}$$

In this case of spheres, we obtain:

Theorem 5. *Given a boundary condition $\mathbf{u}^* \in C^\infty(\partial\Omega)$ satisfying* (21) *there exists a constant $K[R,S]$ depending only on R,S so that if $h < 1$, the unique solution $\mathbf{u} \in V$ to the associated Stokes problem* (1)–(4)(+(6)), *satisfies:*

$$\begin{aligned}
\|\mathbf{u}; V\|^2 &= 6\pi |u^*_\perp(\mathbf{0})|^2 \left[\frac{R_1^2}{h} + \left(\frac{16R_1}{5} - \frac{8R_1^3}{RS} - \frac{3R_1^4}{R_3^3} \right) |\ln(h)| \right] \\
&\quad + \left(2\pi R_1 |\mathbf{u}^*_{/\!/}(\mathbf{0})|^2 + \frac{24\pi}{5} R_1 |R_1 \nabla_{x,y} u^*_\perp(\mathbf{0}) + \frac{(S-R)}{2(R+S)} \mathbf{u}^*_{/\!/}(\mathbf{0})|^2 \right) |\ln(h)| \\
&\quad + K[R,S] (\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2 + \|\mathbf{u}^*; H^{\frac{1}{2}}, \partial\Omega\|^2) \,.
\end{aligned}$$

The norms that we compute in this theorem are related to the forces and torques exerted by the fluid flow on the spheres. Indeed, assume that the sphere $\mathcal{B}$ is moving with a rigid velocity and that we want to compute the forces and torques exerted by a Stokes flow on $\mathcal{B}$ when the distance to the other sphere $\mathcal{B}^*$ is small. By symmetry, we can assume that $\mathcal{B}^*$ is moving with a rigid velocity $\mathbf{u}^*$ and compute the force and torque exerted on $\mathcal{B}^*$. Then, we recall that the set of rigid velocities is a 6-dimensional vector space whose elements are characterized by a translation and angular velocity (computed with respect to the origin for simplicity). Let denote by $(\mathbf{U}_{\to,i}, P_{\to,i})_{i=1,2,3}$ (resp. $(\mathbf{U}_{\circlearrowleft,i}, P_{\circlearrowleft,i})_{i=1,2,3}$) the solutions to the Stokes problem on $\mathcal{F}$ with elementary translational (resp. rotational) boundary conditions

$$\begin{aligned}
\mathbf{U}_{\to,i} &= \mathbf{e}_i\,, && \text{on } \partial\mathcal{B}^*\,,\\
\mathbf{U}_{\to,i} &= \mathbf{0}\,, && \text{on } \partial\mathcal{B}\,,
\end{aligned}
\qquad
\left(\text{resp.} \quad \begin{aligned}
\mathbf{U}_{\circlearrowleft,i} &= \mathbf{e}_i \times \mathbf{x}\,, && \text{on } \partial\mathcal{B}^*\,,\\
\mathbf{U}_{\circlearrowleft,i} &= \mathbf{0}\,, && \text{on } \partial\mathcal{B}\,,
\end{aligned} \right) \qquad \text{for } i = 1,2,3\,.$$

Due to the linearity of the Stokes problem (1)–(4)(+(6)), the solution $(\mathbf{u}, p)$ associated with boundary condition $\mathbf{u}^*$ is then a combination of the $(\mathbf{U}_{\to,i}, P_{\to,i})_{i=1,2,3}$ and $(\mathbf{U}_{\circlearrowleft,i}, P_{\circlearrowleft,i})_{i=1,2,3}$.

Furthermore, straightforward integration by parts yield that the j^{th} component of the force exerted by the flow on $\mathcal{B}^*$ reads:

$$
\begin{aligned}
F_j := \int_{\partial \mathcal{B}^*} (2D(\mathbf{u}) - p\mathbb{I}_3)\nu \mathrm{d}\sigma \cdot \mathbf{e}_j &= 2\int_{\mathcal{F}} D(\mathbf{u}) : D(\mathbf{U}_{\rightarrow,j}) \\
&= \int_{\mathcal{F}} \nabla \mathbf{u} : \nabla \mathbf{U}_{\rightarrow,j} \,,
\end{aligned}
$$

(here ν stands for the normal to $\partial\mathcal{B}^*$ pointing towards $\mathcal{B}^*$). If $\mathbf{u}^* = \mathbf{e}_j$, we obtain:

$$
F_j = \int_{\mathcal{F}} |\nabla \mathbf{U}_{\rightarrow,j}|^2 \,,
$$

and the asymptotic behavior or F_j when $h \to 0$ yields by applying the above theorem. By standard algebraic formulas this identity is generalized to all components of the forces and torques associated with any rigid velocity $\mathbf{u}^*$. Then, one can see the asymptotic expansions we compute in Theorem 5 as a justification and an improvement of the asymptotic values for the matrix $\mathcal{K}$ provided in [6, Section 7] (see [7.6]).

1.3 Two approximation criterions

We conclude this section by providing two criterions which will enable us to measure the distance between an approximation $\mathbf{v}$ and the exact solution $\mathbf{u}$ of the Stokes problem (1)–(4)(+(6)).

First, we recall that one way to prove the existence part of Theorem 3 is to construct $\mathbf{u}_{bdy} \in V[\mathbf{u}^*]$ and to remark that we have $V[\mathbf{u}^*] = \mathbf{u}_{bdy} + V_0$. Existence and uniqueness of a generalized solution then yields from an application of the Stampacchia theorem. This proof entails the expected result together with the following variational characterization:

Proposition 6. *Given $\mathbf{u}^* \in C^\infty(\partial\Omega)$ satisfying (21), the solution $\mathbf{u}$ to (1)–(4)(+(6)) realizes:*

$$
\|\mathbf{u}; V\|^2 = \min \left\{ \int_{\mathcal{F}} |\nabla \mathbf{v}|^2 \,, \quad \mathbf{v} \in V[\mathbf{u}^*] \right\} .
$$

As a consequence, given $\mathbf{u}^* \in C^\infty(\partial\Omega)$, and $\mathbf{u} \in V$ the generalized solution to (1)–(4)(+(6)), we have conversely that any $\mathbf{v} \in V[\mathbf{u}^*]$ satisfies:

$$
\|\mathbf{v}\,; V\| \geq \|\mathbf{u}\,; V\| \,. \tag{23}
$$

This enables to compute many bounds from above for $\|\mathbf{u}; V\|$ by choosing approximations $\mathbf{v}$. The more relevant the approximation, the sharper the bound. To control the distance between an approximation $\mathbf{v}$ and the exact solution $\mathbf{u}$ we have more precisely:

Proposition 7. *Let $\mathbf{u}^* \in C^\infty(\partial\Omega)$ satisfy (21) and denote by $\mathbf{u} \in V$ the solution to (1)–(4)(+(6)). Given $(\mathbf{v}, q) \in V[\mathbf{u}^*] \times C^\infty(\mathcal{F})$ we denote:*

$$
C[\mathbf{v}, q] := \|\Delta \mathbf{v} - \nabla q; V_0^*\| \,,
$$

i.e. *$C[\mathbf{v}, q]$ is the best constant C such that:*

$$
\left| {}_{\mathcal{D}'(\mathcal{F})}\langle (\Delta \mathbf{v} - \nabla q) \,, \mathbf{w} \rangle_{\mathcal{D}(\mathcal{F})} \right| \leq C \|\mathbf{w}\,; V\| \,, \quad \forall\, \mathbf{w} \in \mathcal{V}_0 \,. \tag{24}
$$

Then, there holds:

$$
\|\mathbf{v} - \mathbf{u}\,; V\| \leq C[\mathbf{v}, q] \,. \tag{25}
$$

Proof. The proof is standard. For completeness, we recall it briefly. There holds:

$$
\begin{aligned}
\|\mathbf{v} - \mathbf{u}\,; V\|^2 &= \int_{\mathcal{F}} |\nabla \mathbf{v} - \nabla \mathbf{u}|^2 \\
&= \int_{\mathcal{F}} \nabla \mathbf{v} : \nabla (\mathbf{v} - \mathbf{u}) - \int_{\mathcal{F}} \nabla \mathbf{u} : \nabla (\mathbf{v} - \mathbf{u}) \,.
\end{aligned}
$$

Applying (20) to $\mathbf{w} := \mathbf{v} - \mathbf{u} \in V_0$, we obtain:

$$\int_{\mathcal{F}} \nabla \mathbf{u} : \nabla(\mathbf{v} - \mathbf{u}) = 0\,.$$

Introducing then a sequence $\mathbf{w}_n \in \mathcal{V}_0$ such that $\lim_{n\to\infty} \mathbf{w}_n = \mathbf{w}$ in V, we have:

$$\|\mathbf{v} - \mathbf{u}\,;V\|^2 = \lim_{n\to\infty} \int_{\mathcal{F}} \nabla \mathbf{v} : \nabla \mathbf{w}_n\,,$$

where, for all $n \in \mathbb{N}$, there holds:

$$\int_{\mathcal{F}} \nabla \mathbf{v} : \nabla \mathbf{w}_n = \int_{\mathcal{F}} (\nabla \mathbf{v} - q\mathbb{I}_3) : \nabla \mathbf{w}_n = -\langle (\Delta \mathbf{v} - \nabla q), \mathbf{w}_n \rangle\,.$$

Again, as $\mathbf{w}_n \in V_0$, we apply definition (24) of $C[\mathbf{v}, q]$. This entails:

$$\left| \int_{\mathcal{F}} \nabla \mathbf{v} : \nabla \mathbf{w}_n \right| \leq C[\mathbf{v}, q]\,\|\mathbf{w}_n; V\|\,,$$

and, in the limit $n \to \infty$:

$$\|\mathbf{v} - \mathbf{u}\,;V\|^2 \leq C[\mathbf{v}, q]\|\mathbf{v} - \mathbf{u}\,;V\|\,.$$

This ends the proof. $\square$

We emphasize that in the statement of this criterion there is no geometrical constant in front of $C[\mathbf{v}, q]$ in (25). This is particularly important as we want to consider the influence of the geometry on the relevance of the approximation $\mathbf{v}$. Actually, these geometrical dependencies are hidden in the computation of $C[\mathbf{v}, q]$.

2 Preliminary results on the lubrication problem

In this section, we consider the two-dimensional divergence problem:

$$-\frac{1}{12}\mathrm{div}\left[\gamma^3 \nabla \varpi\right] = f\,, \quad \text{on } B(\mathbf{0}, L)\,, \tag{26}$$

$$\varpi(x, y) = 0\,, \quad \text{on } \partial B(\mathbf{0}, L). \tag{27}$$

We restrict to source terms $f \in C^\infty(\overline{B(\mathbf{0}, L)})$ which have the special form:

$$f = w^* - \frac{1}{2}\mathrm{div}\left((\gamma_t + \gamma_b)\mathbf{v}^*\right)\,, \qquad \text{where } (\mathbf{v}^*, w^*) \in C^\infty(\overline{B(\mathbf{0}, L)}; \mathbb{R}^2 \times \mathbb{R})\,. \tag{28}$$

The weight $\gamma \in C^\infty(B(\mathbf{0}, L))$ is computed with respect to γ_t and γ_b:

$$\gamma(x, y) = h + \gamma_t(x, y) - \gamma_b(x, y)\,, \quad \forall\,(x, y) \in B(\mathbf{0}, L),$$

with a small but positive distance h. The assumptions (A1)–(A3) are in force. It is standard that, since γ is smooth and bounded from below by a strictly positive constant, there exists a unique smooth solution ϖ to (26)-(27). Our aim is to compute estimates on quantities of the form:

$$\int_{B(\mathbf{0}, L)} \gamma^n |\nabla^k \varpi|^2\,, \quad n \in \mathbb{N}\,, k \in \mathbb{N}\,.$$

with explicit dependencies in the data f, the distance h, γ_t and γ_b. In these estimates will appear constants depending directly on $\gamma, \gamma_t, \gamma_b$. We state the definition of these constants as a lemma:

Lemma 8. *There exist constants* $(C_{cvx}, C_{ell}) \in (0,\infty)$ *depending on* γ_t *and* γ_b *only and* $(C_k^{reg})_{k\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ *such that there holds:*

$$\|\gamma_t\,;\,C^k(\overline{B(\mathbf{0},L)})\| + \|\gamma_b\,;\,C^k(\overline{B(\mathbf{0},L)})\| \le C_k^{reg}\,, \qquad \forall\, k\in\mathbb{N}\,, \tag{29}$$

$$C_{cvx} \le \Delta\gamma(x,y)\,, \qquad \forall\,(x,y)\in B(\mathbf{0},L)\,, \tag{30}$$

and, for $h<1$*:*

$$h + C_{ell}(x^2+y^2) \le \gamma(x,y)\,, \qquad \forall\,(x,y)\in B(\mathbf{0},L)\,. \tag{31}$$

The proof of this lemma is an obvious consequence of (A1)-(A3) and is left to the reader. We remark that the constants C_{cvx} and C_{ell} are related to the constant R_1 appearing in assumption (A2). Assumption (A1) together with (31) imply there exists a constant K depending on C_{ell} and C_2^{reg} such that:

$$\begin{aligned} |\nabla\gamma_t(x,y)| + |\nabla\gamma_b(x,y)| &\le K|\gamma(x,y)|^{\frac{1}{2}}\,, \quad \forall\,(x,y)\in B(\mathbf{0},L)\,, \qquad (32)\\ \gamma_t(x,y)+\gamma_b(x,y) &\le K\gamma(x,y)\,, \qquad \forall\,(x,y)\in B(\mathbf{0},L)\,. \qquad (33) \end{aligned}$$

With these conventions, the main results of this section are the following propositions. We first write two weighted first-order estimates on the solution ϖ of (26)-(27) depending on the behavior of the data $\mathbf{v}^*$ and w^* (recall that f is given by (28)) in the origin.

Proposition 9. *Assume that the data* $\mathbf{v}^*, w^*$ *satisfy:*

$$\mathbf{v}^*(\mathbf{0}) = \mathbf{0}\,, \quad w^*(\mathbf{0}) = 0, \quad \nabla w^*(\mathbf{0}) = \mathbf{0}\,. \tag{34}$$

Let $n\in[0,\infty)$*. There exists constant* K_n *depending on* $C_{cvx}, C_{ell}, C_2^{reg}$ *and* n *for which:*

$$\int_{B(\mathbf{0},L)} |\gamma|^{\frac{5}{2}+n}|\nabla\varpi|^2 \le K_n\left\{\|\mathbf{v}^*;H^2(B(0,L))\|^2 + \|w^*;H^3(B(\mathbf{0},L))\|^2\right\}\,, \tag{35}$$

Proposition 10. *Let* $n\in[0,\infty)$*. There exists constant* K_n *depending on* $C_{cvx}, C_{ell}, C_2^{reg}$ *and* n *for which:*

- *if* $n=0$ *we have:*

$$\int_{B(\mathbf{0},L)} \gamma^{3+n}|\nabla\varpi|^2 \le \frac{K_0}{h}\left\{\|\mathbf{v}^*;H^2(B(\mathbf{0},L))\|^2 + \|w^*;H^2(B(\mathbf{0},L))\|^2\right\}, \tag{36}$$

- *if* $n\in(0,1)$ *there holds:*

$$\int_{B(\mathbf{0},L)} \gamma^{3+n}|\nabla\varpi|^2 \le K_n\left\{\frac{|w^*(\mathbf{0})|^2}{h^{1-n}} + \|\mathbf{v}^*;H^2(B(\mathbf{0},L))\|^2 + \|w^*;H^2(B(\mathbf{0},L))\|^2\right\}, \tag{37}$$

- *if* $n=1$ *we have:*

$$\int_{B(\mathbf{0},L)} \gamma^{4}|\nabla\varpi|^2 \le K_1\left\{|w^*(\mathbf{0})|^2|\ln(h)| + \|\mathbf{v}^*;L^2(B(\mathbf{0},L))\|^2 + \|w^*;H^2(B(\mathbf{0},L))\|^2\right\} \tag{38}$$

- *if* $n>1$*, there holds:*

$$\int_{B(\mathbf{0},L)} |\gamma|^{3+n}|\nabla\varpi|^2 \le K_n\left\{\|\mathbf{v}^*;L^2(B(\mathbf{0},L))\|^2 + \|w^*;H^1(B(\mathbf{0},L))\|^2\right\} \tag{39}$$

We emphasize that the constants K_n do not depend on h. In particular, the latter proposition implies that, when $w^*(\mathbf{0})$ vanishes, all quantities:

$$\int_{B(\mathbf{0},L)} |\gamma|^{3+n} |\nabla \varpi|^2, \quad n \in \mathbb{N},$$

remain bounded when h goes to 0. We complement the study with higher order estimates away and around the singularity point $(x, y) = (0, 0)$:

Proposition 11. *Given a integer $k \geq 0$ and $0 < \varepsilon < L$, there exists a positive constant K_k^{reg} depending only on $L, \varepsilon, k, C_{ell}, C_{cvx}, C_{k'}^{reg}$, with $k' = \max(k+1, 2)$, for which ϖ satisfies:*

$$\|\varpi; H^{k+2}(B(\mathbf{0}, L) \setminus B(\mathbf{0}, \varepsilon))\| \leq K_k^{reg} \left[\|\mathbf{v}^*; H^{k+1}(B(\mathbf{0}, L))\| + \|w^*; H^{k'-1}(B(\mathbf{0}, L))\| \right]. \quad (40)$$

Proposition 12. *Let $k \in \{0, \ldots, 6\}$, $n \in (k, \infty)$ and denote $e_k := \max(k, 3)$. There exists a constant $K_{n,k}^{sing}$ depending on $k, n, C_{ell}, C_{cvx}, C_{k'}^{reg}$, with $k' = \max(k+1, 2)$, such that:*

$$\int_{B(\mathbf{0},L)} |\gamma|^{3+n} |\nabla^{k+1} \varpi|^2 \leq K_{n,k}^{sing} \Big\{ \int_{B(\mathbf{0},L)} |\gamma|^{3+(n-k)} |\nabla \varpi|^2 + \|\mathbf{v}^*; H^{k+1}(B(\mathbf{0}, L))\|^2 + \|w^*; H^{e_k}(B(\mathbf{0}, L))\|^2 \Big\}. \quad (41)$$

In the remainder of this section, we first give proofs for these propositions. We then conclude by showing that the first order estimates are optimal in the case of sphere.

2.1 First order estimates : proofs of propositions 9 and 10

System (26)-(27) is associated with the weak formulation:

$$\frac{1}{12} \int_{B(\mathbf{0},L)} \gamma^3 \nabla \varpi \cdot \nabla \varphi = \int_{B(\mathbf{0},L)} f \varphi, \quad \forall \varphi \in C_c^\infty(B(\mathbf{0}, L)). \quad (42)$$

Hence, to construct solutions, one introduces the bilinear form:

$$(\varphi, \psi) \mapsto ((\varphi, \psi))_\gamma := \frac{1}{12} \int_{B(\mathbf{0},L)} \gamma^3 \nabla \varphi \cdot \nabla \psi.$$

A definition for weak solution to (26)-(27) then reads:

Definition 13. *We call weak solution to (26)-(27) any $\varpi \in H_0^1(B(\mathbf{0}, L))$ such that:*

$$((\varpi, \varphi))_\gamma = \int_{B(\mathbf{0},L)} f \varphi, \quad \forall \varphi \in H_0^1(B(\mathbf{0}, L)).$$

Thanks to (29)-(31), $((\cdot, \cdot))_\gamma$ defines a coercive continuous bilinear form on $H_0^1(B(\mathbf{0}, L))$. Existence and uniqueness of a weak solution to (26)-(27) yields as a straightforward application of the Stampacchia theorem. Given our regularity assumptions on the distance function γ, classical results on divergence problems apply so that this weak solution is smooth on $\overline{B(\mathbf{0}, L)}$ and satisfies (26)-(27) in a classical sense (see [10, Chapter 8]). All the estimates in [10, Chapter 8] depend *a priori* deeply on h so that an alternative approach is required.

In the proofs below, we denote with K a constant which is important to our computations and shall put in brackets its relevant parameters (such as $(C_{cvx}, C_{ell}), (C_k^{reg})_{k \in \mathbb{N}}$). These constants may differ from line to line. Most of these constants shall also depend on the parameter L but we include this tacitly. Notations C are used for generic constants that depend only on L or on other parameters that are irrelevant to our computations. Again, they may differ from line to line. Regarding constants K, it might appear that a constant depend on C_k^{reg} and C_j^{reg} for $k \neq j$. In this case, we have that the constant K depends on the k-first derivatives of γ_t and γ_b and also on the j-first derivatives of γ_t and γ_b. We shall simplify K then into a function depending only on the l-first derivatives of γ_t and γ_b with $l = \max(k,j)$, *i.e.*, on C_l^{reg}.

From now on, we fix data $(\mathbf{v}^*, w^*)$ and denote the associated (weak) solution ϖ. The first step of the proofs is the following preliminary lemma:

Lemma 14. *Given $\alpha \in [-1,2)$, there exists a constant $K_\alpha := K[C_{cvx}, C_2^{reg}, C_{ell}, \alpha]$ s.t. ϖ satisfies:*

$$\int_{B(\mathbf{0},L)} |\gamma|^{3+\alpha}|\nabla\varpi|^2 \leq K_\alpha \int_{B(\mathbf{0},L)} \left[\gamma^{\alpha-1}|\mathbf{v}^*|^2 + \gamma^{\alpha-2}|w^*|^2\right] \tag{43}$$

Proof. Given $\alpha \geq -1$, as γ is a positive function, explicit computations yield that:

$$\Delta[\gamma^{\alpha+3}] = (\alpha+3)[\Delta\gamma]\gamma^{\alpha+2} + (\alpha+3)(\alpha+2)|\nabla\gamma|^2\gamma^{\alpha+1}\,.$$

Introducing the bound (30) on $\Delta\gamma$, we deduce that:

$$|\nabla\gamma|^2\gamma^{\alpha+1} \leq \frac{1}{(\alpha+3)(\alpha+2)}\Delta[\gamma^{\alpha+3}]\,, \tag{44}$$

$$\gamma^{\alpha+2} \leq \frac{1}{2C_{cvx}}\Delta[\gamma^{\alpha+3}]\,. \tag{45}$$

Now, for any $\varphi \in C^\infty(\overline{B(\mathbf{0},L)})$, integrating by parts and applying Hölder inequality yields:

$$\begin{aligned} 0 \leq \int_{B(\mathbf{0},L)} \Delta[\gamma^{\alpha+3}]|\varphi|^2 &= -2(\alpha+3)\int_{B(\mathbf{0},L)} \gamma^{\alpha+2}\nabla\gamma\cdot\varphi\nabla\varphi + \int_{\partial B(\mathbf{0},L)} |\varphi|^2\partial_\nu\gamma^{\alpha+3}\mathrm{d}\sigma \\ &\leq (\alpha+3)\left[\frac{\alpha+2}{2}\int_{B(\mathbf{0},L)} \gamma^{\alpha+1}|\nabla\gamma|^2\varphi^2 + \frac{2}{\alpha+2}\int_{B(\mathbf{0},L)} \gamma^{\alpha+3}|\nabla\varphi|^2\right] \\ &\quad + K[C_1^{reg},\alpha]\int_{\partial B(\mathbf{0},L)} |\varphi|^2\mathrm{d}\sigma\,. \end{aligned}$$

Applying then (44) to bound the first term in the right-hand side of this last inequality, we obtain:

$$\left|\int_{B(\mathbf{0},L)} \Delta[\gamma^{\alpha+3}]|\varphi|^2\right| \leq \frac{4(\alpha+3)}{\alpha+2}\int_{B(\mathbf{0},L)} \gamma^{\alpha+3}|\nabla\varphi|^2 + K[C_1^{reg},\alpha]\int_{\partial B(\mathbf{0},L)} |\varphi|^2\mathrm{d}\sigma. \tag{46}$$

We multiply now (26) with $\gamma^\alpha\varpi$. This yields:

$$\begin{aligned} \int_{B(\mathbf{0},L)} \gamma^{\alpha+3}|\nabla\varpi|^2 - \frac{\alpha}{2(\alpha+3)}\int_{B(\mathbf{0},L)} \Delta[\gamma^{\alpha+3}]|\varpi|^2 & \\ = 12\int_{B(\mathbf{0},L)} \left(w^* - \frac{1}{2}\mathrm{div}[(\gamma_t+\gamma_b)\mathbf{v}^*]\right)\gamma^\alpha\varpi\,. & \end{aligned} \tag{47}$$

We compute separately the *RHS* and the *LHS* of this identity. If $\alpha < 0$ the last term on the RHS is positive. If $\alpha \in [0,2]$, we remark that ϖ vanishes on $\partial B(\mathbf{0},L)$ and apply (46) to ϖ. This entails:

$$LHS \geq \left(1 - \frac{2\max(\alpha,0)}{(\alpha+2)}\right)\int_{B(\mathbf{0},L)} \gamma^{\alpha+3}|\nabla\varpi|^2. \tag{48}$$

We note here that, since $\alpha < 2$ the factor appearing in the right-hand side of this identity is positive. The fact that this factor changes sign when α crosses 2 makes this proof irrelevant for $\alpha \geq 2$.

Concerning the *RHS*, we have first, applying (45) and (46) (with $\varphi = \varpi$ and noting that $4(\alpha+3)/(\alpha+2) \leq 8$), that, for arbitrary $\varepsilon > 0$, there holds:

$$\left|12 \int_{B(\mathbf{0},L)} w^* \gamma^\alpha \varpi\right| \leq \frac{K[C_{cvx}]}{\varepsilon} \int_{B(\mathbf{0},L)} |w^*|^2 \gamma^{\alpha-2} + \frac{\varepsilon}{8} \int_{B(\mathbf{0},L)} \gamma^{\alpha+3} |\nabla \varpi|^2 .$$

Similarly, by applying (44), (45) and (46) (again with $\varphi = \varpi$) and (33), we obtain that, for $\alpha \in [-1, 2)$ and arbitrary $\varepsilon > 0$, there holds:

$$\begin{aligned}
\left|\int_{B(\mathbf{0},L)} \mathrm{div}\,(\gamma_t + \gamma_b)\mathbf{v}^* \gamma^\alpha \varpi\right| &\leq \left|\alpha \int_{B(\mathbf{0},L)} (\gamma_t + \gamma_b)\mathbf{v}^* \cdot \nabla\gamma\, \gamma^{\alpha-1} \varpi\right| + \left|\int_{B(\mathbf{0},L)} (\gamma_t + \gamma_b)\mathbf{v}^* \cdot \nabla\varpi \gamma^\alpha\right| \\
&\leq \frac{K[C_{ell}, C_2^{reg}]}{\varepsilon} \int_{B(\mathbf{0},L)} |\mathbf{v}^*|^2 \gamma^{\alpha-1} + \varepsilon \int_{B(\mathbf{0},L)} \gamma^{\alpha+1} |\nabla\gamma|^2 |\varpi|^2 + \varepsilon \int_{B(\mathbf{0},L)} \gamma^{\alpha+3} |\nabla\varpi|^2 \\
&\leq \frac{K[C_{ell}, C_2^{reg}]}{\varepsilon} \int_{B(\mathbf{0},L)} |\mathbf{v}^*|^2 \gamma^{\alpha-1} + \varepsilon \int_{B(\mathbf{0},L)} \Delta[\gamma^{\alpha+3}] |\varpi|^2 + \varepsilon \int_{B(\mathbf{0},L)} \gamma^{\alpha+3} |\nabla\varpi|^2 \\
&\leq \frac{K[C_{ell}, C_2^{reg}]}{\varepsilon} \int_{B(\mathbf{0},L)} |\mathbf{v}^*|^2 \gamma^{\alpha-1} + \frac{\varepsilon}{4} \int_{B(\mathbf{0},L)} \gamma^{\alpha+3} |\nabla\varpi|^2 .
\end{aligned}$$

This yields finally that we have, for arbitrary $\varepsilon > 0$:

$$\begin{aligned}
RHS \leq \frac{K[C_{ell}, C_2^{reg}, C_{cvx}]}{\varepsilon} &\left(\int_{B(\mathbf{0},L)} |\mathbf{v}^*|^2 \gamma^{\alpha-1} + \int_{B(\mathbf{0},L)} |w^*|^2 \gamma^{\alpha-2}\right) \\
&+ \frac{\varepsilon}{4} \int_{B(\mathbf{0},L)} \gamma^{\alpha+3} |\nabla\varpi|^2 . \qquad (49)
\end{aligned}$$

We replace finally the right-hand side and left-hand side of (47) with (49) and (48) and obtain the expected result by choosing ε sufficiently small depending on α. □

We are now in position to prove Proposition 9 and Proposition 10.

Proof of Proposition 9. Because γ is bounded by C_0^{reg} on $\overline{B(\mathbf{0},L)}$, it is sufficient to prove (35) in the case $n = 0$. Under the further assumption (34), there holds (from the two-dimensional embedding $H^{m+2}(B(\mathbf{0},L)) \subset C^{m,3/4}(B(\mathbf{0},L)))$ for arbitrary $m \in \mathbb{N} \cup \{0\}$)

$$\begin{aligned}
|\mathbf{v}^*(x,y)| &\leq C(|x|^2 + |y|^2)^{\frac{3}{8}} \|\mathbf{v}^*; H^2(B(\mathbf{0},L))\| , &(50)\\
|w^*(x,y)| &\leq C(|x|^2 + |y|^2)^{\frac{7}{8}} \|w^*; H^3(B(\mathbf{0},L))\| , &(51)
\end{aligned}$$

for $(x,y) \in B(\mathbf{0},L)$.

Applying then Lemma 14 to ϖ with $\alpha = -1/2$, we obtain:

$$\int_{B(\mathbf{0},L)} \gamma^{\frac{5}{2}} |\nabla\varpi|^2 \leq K[C_{ell}, C_{cvx}, C_2^{reg}] \left(\int_{B(\mathbf{0},L)} \frac{|\mathbf{v}^*|^2}{\gamma^{\frac{3}{2}}} + \frac{|w^*|^2}{\gamma^{\frac{5}{2}}}\right)$$

Here we call (50)-(51) and (31) to obtain:

$$\begin{aligned}
\int_{B(\mathbf{0},L)} \frac{|\mathbf{v}^*|^2}{\gamma^{\frac{3}{2}}} + \frac{|w^*|^2}{\gamma^{\frac{5}{2}}} &\leq C \int_0^L \left\{ \frac{r^{\frac{3}{2}} \|\mathbf{v}^*; H^2(B(\mathbf{0},L))\|^2}{C_{ell}^{3/2} r^3} + \frac{r^{\frac{7}{2}} \|w^*; H^3(B(\mathbf{0},L))\|^2}{C_{ell}^{5/2} r^5} \right\} r\mathrm{d}r \\
&\leq K[C_{ell}] \left\{ \|\mathbf{v}^*; H^2(B(\mathbf{0},L))\|^2 + \|w^*; H^3(B(\mathbf{0},L))\|^2 \right\} .
\end{aligned}$$

This ends the proof. □

Proof of Proposition 10. Because γ is bounded by C_0^{reg} on $\overline{B(\mathbf{0},L)}$, it is sufficient to prove (39) and (38) in the cases $n \in [0,1)$, $n = 1$ and $n \in (1,2)$.

Case $n \in (1,2)$ Applying Lemma 14 with $\alpha = n$, we have, with a constant K_n depending on $C_{cvx}, C_{ell}, C_2^{reg}$ and n:

$$\int_{B(\mathbf{0},L)} |\gamma|^{3+n} |\nabla \varpi|^2 \quad \leq \quad K_n \left[\int_{B(\mathbf{0},L)} \gamma^{n-1} |\mathbf{v}^*|^2 + \gamma^{n-2} |w^*|^2 \right]$$

Here, we note that $n > 1$ so that $2-n < 1$. Hence, we might fix $p_n \in (1, \infty)$ such that $p_n(2-n) < 1$. Introducing q_n the conjugate exponent, we get:

$$\int_{B(\mathbf{0},L)} \gamma^{n-2} |w^*|^2 \leq \left(\int_{B(\mathbf{0},L)} \frac{1}{\gamma^{(2-n)p_n}} \right)^{\frac{1}{p_n}} \|w^*; L^{2q_n}(B(\mathbf{0},L))\|^2 \,.$$

In the right-hand side of this last line, we note that standard 2D-Sobolev imbeddings yield

$$\|w^*; L^{2q_n}(B(\mathbf{0},L))\|^2 \leq C_n \|w^*; H^1(B(\mathbf{0},L))\|^2$$

and that, from (31):

$$\int_{B(\mathbf{0},L)} \frac{1}{\gamma^{(2-n)p_n}} \leq K[C_{ell}, n] \int_0^L \frac{r\mathrm{d}r}{r^{2(2-n)p_n}} \leq K[C_{ell}, L, n],$$

as $2(2-n)p_n < 2$. This ends up the proof in the last case.

Case $n = 1$ Applying the embedding $H^2(B(\mathbf{0},L)) \subset C^{0,3/4}(B(\mathbf{0},L))$, we have:

$$|w^*(x,y)| \leq |w^*(\mathbf{0})| + C(x^2+y^2)^{\frac{3}{8}} \|w^*; H^2(B(\mathbf{0},L))\| \,, \quad \forall (x,y) \in B(\mathbf{0},L).$$

The remainder of the computation follows the line of the previous case. Applying Lemma 14 with $\alpha = 1$, we have, with a constant K depending on the same quantities:

$$\begin{aligned}
\int_{B(\mathbf{0},L)} |\gamma|^4 |\nabla \varpi|^2 &\leq K_1 \left[\int_{B(\mathbf{0},L)} |\mathbf{v}^*|^2 + \frac{|w^*|^2}{\gamma} \right] \\
&\leq \quad K \left[\|\mathbf{v}^*; L^2(B(\mathbf{0},L))\|^2 + \int_{B(\mathbf{0},L)} \frac{|w^*(\mathbf{0})|^2}{\gamma} + \int_{B(\mathbf{0},L)} \frac{r^{\frac{3}{2}}}{\gamma} \|w^*; H^2(B(\mathbf{0},L))\|^2 \right], \\
&\leq \quad K \left[\|\mathbf{v}^*; L^2(B(\mathbf{0},L))\|^2 + \int_0^L \frac{|w^*(\mathbf{0})|^2 r\mathrm{d}r}{h + C_{ell} r^2} + \|w^*; H^2(B(\mathbf{0},L))\|^2 \right], \\
&\leq \quad K \left[\|\mathbf{v}^*; L^2(B(\mathbf{0},L))\|^2 + |w^*(\mathbf{0})|^2 |\ln(h)| + \|w^*; H^2(B(\mathbf{0},L))\|^2 \right].
\end{aligned}$$

Case $n \in (0,1)$ Again, we apply Lemma 14 with $\alpha = n$, and apply the embedding $H^2(B(\mathbf{0},L)) \subset C^{0,1-n/2}(B(\mathbf{0},L))$. This yields with similar computations as previously:

$$\begin{aligned}
&\int_{B(\mathbf{0},L)} |\gamma|^{3+n} |\nabla \varpi|^2 \leq K_n \left[\int_{B(\mathbf{0},L)} \frac{|\mathbf{v}^*|^2}{\gamma^{1-n}} + \frac{|w^*|^2}{\gamma^{2-n}} \right] \\
&\leq \quad K[C_0^{reg}, C_{ell}, n] \left[\int_0^L \frac{\|\mathbf{v}^*; H^2(B(\mathbf{0},L))\|^2 r\mathrm{d}r}{\gamma^{1-n}} + \int_0^L \frac{|w^*(0)|^2 r + r^{3-n} \|w^*; H^2(B(\mathbf{0},L))\|^2}{\gamma^{2-n}} \mathrm{d}r \right], \\
&\leq \quad K[C_0^{reg}, C_{ell}, n] \left[\|\mathbf{v}^*; H^2(B(\mathbf{0},L))\|^2 + \frac{|w^*(0)|^2}{h^{1-n}} + \|w^*; H^2(B(\mathbf{0},L))\|^2 \right].
\end{aligned}$$

Case $n = 0$ We conclude with similar arguments, applying 14 with $\alpha = 0$, and the embedding $H^2(B(\mathbf{0}, L)) \subset C^0(B(\mathbf{0}, L))$. This ends the proof. $\square$

2.2 Higher order estimates: proofs of propositions 11 and 12

We compute now estimates involving $\nabla^k \varpi$ with $k \geq 1$. We start with estimates away from the origin.

Proof of Proposition 11. This proposition follows from the classical regularity theory for elliptic problems, as developed in [10]. Indeed, fix $k \geq 0$ and $0 < \varepsilon < L < \infty$. Applying Proposition 10 with $n = 2$, and combining with (31), we obtain for a constant K wich depends on $C_{cvx}, C_{ell}, C_2^{reg}$ and L that

$$\begin{aligned}[C_{ell}\,\varepsilon^2]^5 \int_{B(\mathbf{0},L)\setminus B(0,\varepsilon)} |\nabla \varpi|^2 &\leq \int_{B(\mathbf{0},L)} |\gamma|^5 |\nabla \varpi|^2 \\ &\leq K\left[\|\mathbf{v}^*; L^2(B(\mathbf{0},L))\|^2 + \|w^*; H^1(B(\mathbf{0},L))\|^2\right]. \end{aligned} \tag{52}$$

As $\varpi = 0$ on $\partial B(\mathbf{0}, L)$, we deduce from this inequality (applying the variant of Poincaré inequality given in [8, Exercicse II.5.13]) that there exists a constant $K := K[\varepsilon, C_{cvx}, C_{ell}, C_2^{reg}, L]$ so that:

$$\|\varpi; H^1(B(\mathbf{0},L) \setminus B(\mathbf{0},\varepsilon))\| \leq K\left[\|\mathbf{v}^*; L^2(B(\mathbf{0},L))\|^2 + \|w^*; H^1(B(\mathbf{0},L))\|^2\right]. \tag{53}$$

Then, we note that the operator $L := \mathrm{div}[\gamma^3 \nabla]$ is uniformly elliptic on $B(\mathbf{0}, L) \setminus B(0, \varepsilon)$ with ellipticity constant $\lambda_\varepsilon := \inf_{B(\mathbf{0},L)\setminus B(0,\varepsilon)} \gamma^3 \geq (C_{ell}\varepsilon^2)^3$ depending only on C_{ell} and ε. Finally, applying [10, Theorem 8.8 p.183 with Theorem 8.12 p.186] if $k = 0$ or [10, Theorem 8.10 p.186 with Theorem 8.13, p.187] if $k \geq 1$ implies that there exists a constant $\tilde{K}_k^{reg}$ depending on ε, C_{ell} and C_{k+1}^{reg} so that:

$$\begin{aligned}&\|\varpi; H^{k+2}(B(\mathbf{0},L) \setminus B(\mathbf{0},\varepsilon))\| \\ &\qquad \leq \tilde{K}_k^{reg}\left[\|\varpi; H^1(B(\mathbf{0},L) \setminus B(\mathbf{0},\varepsilon))\| + \|f; H^k(B(\mathbf{0},L) \setminus B(\mathbf{0},\varepsilon))\|\right], \\ &\qquad \leq K_k^{reg}\left[\|\mathbf{v}^*; H^{k+1}(B(\mathbf{0},L))\|^2 + \|w^*; H^{k'-1}(B(\mathbf{0},L))\|^2\right],\end{aligned}$$

where we applied (53) to reach the last line. $\square$

This proposition is a tool for computing the traces of derivatives of ϖ on $\partial B(\mathbf{0}, L)$. With this consideration, we obtain Proposition 12.

Proof of Proposition 12. To prepare the proof, we mention that the chain rule together with (32) yields that for any integer $i \in \{0, \ldots, 6\}$ there exists a constant $K[i]$ depending on C_i^{reg}, C_{ell} and C_2^{reg}, for which:

$$|\nabla^i \gamma^3(x,y)| \leq K[i] |\gamma(x,y)|^{3-\frac{i}{2}}, \quad \forall\, (x,y) \in B(\mathbf{0},L). \tag{54}$$

This inequality no longer holds when $k > 6$. Again, this is the reason for our lemma to hold only for values of the parameter k below 6.

We prove now by induction on k for fixed $m \in (0, \infty)$ that there holds:

" there exists a constant $K_{m,k}^{sing}$ depending on $k, m, C_{ell}, C_{cvx}, C_{k'}^{reg}$, with $k' = \max(k+1, 2)$, such that, denoting by $e_k = \max(k, 3)$:

$$\begin{aligned}\int_{B(\mathbf{0},L)} |\gamma|^{3+k+m} |\nabla^{k+1}\varpi|^2 \leq K_{m,k}^{sing}\Big\{ \int_{B(\mathbf{0},L)} |\gamma|^{3+m} |\nabla \varpi|^2 &+ \|\mathbf{v}^*; H^{k+1}(B(\mathbf{0},L))\|^2 \\ &+ \|w^*; H^{e_k}(B(\mathbf{0},L))\|^2 \Big\}." \end{aligned} \tag{55}$$

The case $k = 0$ is obvious. To prove the induction argument, we fix $k \in \{0, \dots, 5\}$ and introduce D^{k+1} a (homogeneous) differential operator of order $k+1$ having constant coefficients. Applying D^{k+1} to (26) yields:

$$-\frac{1}{12}\mathrm{div}[\gamma^3 \nabla D^{k+1}\varpi] = \mathrm{div}\left[R^{k+1} - D^{k+1}\frac{1}{2}(\gamma_t + \gamma_b)\mathbf{v}^*\right] + D^{k+1}w^*\,. \tag{56}$$

where:

$$R^{k+1} := \frac{1}{12}\left(D^{k+1}[\gamma^3\nabla\varpi] - \gamma^3\nabla D^{k+1}\varpi\right).$$

We multiply (56) with $\gamma^{k+m+1}D^{k+1}\varpi$. After integration by parts, this yields:

$$\frac{1}{12}\int_{B(\mathbf{0},L)} \gamma^{3+m+(k+1)}|\nabla D^{k+1}\varpi|^2 = \frac{RHS_1}{12} + RHS_2, \tag{57}$$

where:

$$\begin{aligned}
RHS_1 &= \int_{\partial B(\mathbf{0},L)} \gamma^{4+m+k}\partial_\nu D^{k+1}\varpi D^{k+1}\varpi - \int_{B(\mathbf{0},L)} (k+m+1)\nabla\gamma\gamma^{3+m+k}\nabla D^{k+1}\varpi D^{k+1}\varpi \\
RHS_2 &= \int_{B(\mathbf{0},L)} (D^{k+1}w^* - \mathrm{div}D^{k+1}\frac{(\gamma_t+\gamma_b)\mathbf{v}^*}{2})\gamma^{k+m+1}D^{k+1}\varpi \\
&\quad - \int_{B(\mathbf{0},L)} R^{k+1}\cdot(\gamma^{k+m+1}\nabla D^{k+1}\varpi + (k+m+1)\nabla\gamma\gamma^{k+m}D^{k+1}\varpi)\,. \\
&\quad + \int_{\partial B(\mathbf{0},L)} R^{k+1}\cdot\nu\gamma^{k+m+1}D^{k+1}\varpi
\end{aligned}$$

We bound the first term in RHS_1 by applying trace theorems and Proposition 11: there exists a constant $C[D^{k+1}]$ (depending on D^{k+1} and also on L) for which

$$\begin{aligned}
&\int_{\partial B(\mathbf{0},L)} \left[|D^{k+1}\varpi|^2 + |\partial_\nu D^{k+1}\varpi|^2\right]\mathrm{d}\sigma \\
&\qquad \leq C[D^{k+1}]\|\nabla^{k+1}\varpi; H^1(\partial(B(\mathbf{0},L)/B(\mathbf{0},L/2)))\|^2 \\
&\qquad \leq C[D^{k+1}]\|\varpi; H^{k+3}(B(\mathbf{0},L)/(B(\mathbf{0},L/2)))\|^2 \\
&\qquad \leq C[D^{k+1}]K^{reg}_{k+1}\left[\|\mathbf{v}^*; H^{k+2}(B(\mathbf{0},L))\|^2 + \|w^*; H^{k+1}(B(\mathbf{0},L))\|^2\right]\,.
\end{aligned}$$

For the second term, we apply (32) and a Hölder inequality:

$$\begin{aligned}
&\left|\int_{B(\mathbf{0},L)} (k+m+1)\nabla\gamma\gamma^{3+m+k}\nabla D^{k+1}\varpi D^{k+1}\varpi\right| \\
&\qquad \leq K[C_{ell}, C_2^{reg}]\int_{B(\mathbf{0},L)} \gamma^{7/2+m+k}|\nabla D^{k+1}\varpi||D^{k+1}\varpi| \\
&\qquad \leq \frac{1}{2}\int_{B(\mathbf{0},L)} \gamma^{3+m+(k+1)}|\nabla D^{k+1}\varpi|^2 + K[C_{ell}, C_2^{reg}]\int_{B(\mathbf{0},L)} \gamma^{3+m+k}|D^{k+1}\varpi|^2\,.
\end{aligned}$$

To bound the last term, we note that $|D^{k+1}\varpi| \leq C[D^{k+1}]|\nabla^{k+1}\varpi|$. Consequently, applying the induction assumption, we obtain:

$$\begin{aligned}
|RHS_1| \leq C[D^{k+1}]K[C_{ell}, C_2^{reg}, K^{sing}_{m,k}, K^{reg}_{k+1}]\Big\{\int_{B(\mathbf{0},L)} |\gamma|^{3+m}|\nabla\varpi|^2 + \|\mathbf{v}^*; H^{k+2}(B(\mathbf{0},L))\|^2 \\
+ \|w^*; H^{e_{k+1}}(B(\mathbf{0},L))\|^2\Big\} + \frac{1}{2}\int_{B(\mathbf{0},L)} \gamma^{3+m+(k+1)}|\nabla D^{k+1}\varpi|^2\,.
\end{aligned}$$

As for RHS_2, we distinguish between two cases. If $k+m \geq 1$ we have that $2(m+k+1) \geq 3+m+k$ so that a standard Hölder inequality yields, with the induction assumption:

$$
\begin{aligned}
&\left| \int_{B(\mathbf{0},L)} (D^{k+1} w^* - \frac{1}{2} \mathrm{div} D^{k+1}(\gamma_t + \gamma_b)\mathbf{v}^*) \gamma^{k+m+1} D^{k+1} \varpi \right| \\
\leq \quad & C[D^{k+1}] \left[\|w^*; H^{k+1}(B(\mathbf{0},L))\|^2 + C^{reg}_{k+2} \|\mathbf{v}^*; H^{k+2}(B(\mathbf{0},L))\|^2 + \int_{B(\mathbf{0},L)} \gamma^{3+m+k} |\nabla^{k+1} \varpi|^2 \right] \\
\leq \quad & C[D^{k+1}] K[K^{sing}_{m,k}, C^{reg}_{k+2}] \left[\int_{B(\mathbf{0},L)} \gamma^{3+m} |\nabla \varpi|^2 + \|\mathbf{v}^*; H^{k+2}(B(\mathbf{0},L))\|^2 \right. \\
& \qquad \qquad \left. + \|w^*; H^{e_k}(B(\mathbf{0},L))\|^2 \right].
\end{aligned}
$$

Whereas, if $0 < (k+m) < 1$ so that $k = 0$ in particular, we introduce arbitrarily the missing powers of γ and recall (32)-(33):

$$
\begin{aligned}
&\left| \int_{B(\mathbf{0},L)} (D^{k+1} w^* - \frac{1}{2} \mathrm{div} D^{k+1}(\gamma_t + \gamma_b)\mathbf{v}^*) \gamma^{k+m+1} D^{k+1} \varpi \right| \\
& \qquad \leq \quad C \left| \int_{B(\mathbf{0},L)} \left(\frac{|\nabla w^*|}{\gamma^{\frac{1}{2} - \frac{(k+m)}{2}}} + \frac{K[C^{reg}_2, C_{ell}]}{\gamma^{\frac{1}{2} - \frac{(k+m)}{2}}} \sum_{j=0}^{2} \gamma^{\frac{j}{2}} |\nabla^j \mathbf{v}^*| \right) \gamma^{\frac{(k+m)}{2} + \frac{3}{2}} D^{k+1} \varpi \right|
\end{aligned}
$$

where $1 - (k+m) < 1$ so that:

$$
\int_{B(\mathbf{0},L)} \frac{|\nabla w^*|^2}{\gamma^{1-(k+m)}} \leq K[C_{ell}] \|w^*; H^3(B(\mathbf{0},L))\|^2 ,
$$

and

$$
\int_{B(\mathbf{0},L)} \left| \frac{1}{\gamma^{\frac{1}{2} - \frac{(k+m)}{2}}} \sum_{j=0}^{2} \gamma^{\frac{j}{2}} |\nabla^j \mathbf{v}^*| \right|^2 \leq K[C_{ell}, C^{reg}_0] \|\mathbf{v}^*; H^2(B(\mathbf{0},L))\|^2 .
$$

Hence a Hölder inequality yields that:

$$
\begin{aligned}
&\left| \int_{B(\mathbf{0},L)} (D^{k+1} w^* - \frac{1}{2} \mathrm{div} D^{k+1}(\gamma_t + \gamma_b)\mathbf{v}^*) \gamma^{k+m+1} D^{k+1} \varpi \right| \\
&\leq K[K^{sing}_{m,0}, C^{reg}_2, C_{ell}] \left[\int_{B(\mathbf{0},L)} \gamma^{3+m} |\nabla \varpi|^2 + \|\mathbf{v}^*; H^{k+2}(B(\mathbf{0},L))\|^2 + \|w^*; H^{e_k}(B(\mathbf{0},L))\|^2 \right].
\end{aligned}
$$

To treat the last terms, we expand the differential operator R^{k+1} and apply (54) yielding that there exists also a constant $C[D^{k+1}]$ for which:

$$
|R^{k+1}| \leq C[D^{k+1}] \sum_{l=0}^{k} |\nabla^{l+1} \varpi| |\nabla^{k+1-l} \gamma^3| \leq C[D^{k+1}] \sum_{l=0}^{k} K[k+1-l] |\gamma|^{3 - \frac{k+1-l}{2}} |\nabla^{l+1} \varpi| .
$$

Consequently, we bound similarly as above by applying trace theorems and Proposition 11:

$$
\begin{aligned}
\left| \int_{\partial B(\mathbf{0},L)} R^{k+1} \cdot \nu \gamma^{k+m+1} D^{k+1} \varpi \right| &\leq C[D^{k+1}] K[C^{reg}_{k+1}, C_{ell}, C^{reg}_2] \|\varpi; H^{k+1}(\partial B(\mathbf{0},L))\|^2 \\
&\leq \quad C[D^{k+1}] K[C^{reg}_{k+1}, C_{ell}, C^{reg}_2] \|\varpi; H^{k+2}(B(\mathbf{0},L) \setminus B(\mathbf{0},L/2))\|^2 \\
&\leq \quad C[D^{k+1}] K[C^{reg}_{k+1}, C_{ell}, C^{reg}_2] \left[\|\mathbf{v}^*; H^{k+1}(B(\mathbf{0},L))\|^2 + \|w^*; H^{e_k}(B(\mathbf{0},L))\|^2 \right] .
\end{aligned}
$$

For the other term, we recall (32) and apply the induction assumption. This yields:

$$
\begin{aligned}
& \left| \int_{B(\mathbf{0},L)} R^{k+1} \cdot (\gamma^{k+m+1} \nabla D^{k+1} \varpi + (k+m+1) \nabla \gamma \gamma^{k+m} D^{k+1} \varpi) \right| \\
\leq \quad & K[C^{reg}_{k+1}, C_{ell}, C^{reg}_2] \sum_{l=0}^{k} \int_{B(\mathbf{0},L)} \gamma^{\frac{3+m+l}{2}} |\nabla^{l+1} \varpi| \left(\gamma^{\frac{3+m+(k+1)}{2}} |\nabla D^{k+1} \varpi| + \gamma^{\frac{3+m+k}{2}} |\nabla^{k+1} \varpi| \right) \\
\leq \quad & \frac{1}{48} \int_{B(\mathbf{0},L)} \gamma^{3+m+(k+1)} |\nabla D^{k+1} \varpi|^2 \\
& \quad + K[C^{reg}_{k+1}, C_{ell}, C^{reg}_2] K^{sing}_{m,k} \left[\int_{B(\mathbf{0},L)} |\gamma|^{3+m} |\nabla \varpi|^2 + \|\mathbf{v}^*; H^{k+1}(B(\mathbf{0},L))\|^2 \right. \\
& \quad \left. + \|w^*; H^{e_k}(B(\mathbf{0},L))\|^2 \right]
\end{aligned}
$$

We obtain finally that there exists a constant $K^{sing}_{m,k+1}$ depending on $K^{reg}_{k+1}, K^{sing}_{m,k}, K^{sing}_{0,k}$ and $C^{reg}_{k+2}, C^{reg}_2, C_{ell}$ for which:

$$
\begin{aligned}
& \left| \frac{RHS_1}{12} + RHS_2 \right| \\
& \leq C[D^{k+1}] K^{sing}_{m,k+1} \left[\int_{B(\mathbf{0},L)} |\gamma|^{3+m} |\nabla \varpi|^2 + \|\mathbf{v}^*; H^{k+2}(B(\mathbf{0},L))\|^2 + \|w^*; H^{e_{k+1}}(B(\mathbf{0},L))\|^2 \right] \\
& \qquad + \frac{3}{48} \int_{B(\mathbf{0},L)} \gamma^{3+m+(k+1)} |\nabla D^{k+1} \varpi|^2 .
\end{aligned}
$$

We introduce this inequality into (57) and obtain that:

$$
\begin{aligned}
& \int_{B(\mathbf{0},L)} |\gamma|^{3+m+(k+1)} |\nabla D^{k+1} \varpi|^2 \\
& \leq C[D^{k+1}] K^{sing}_{m,k+1} \left[\int_{B(\mathbf{0},L)} |\gamma|^{3+m} |\nabla \varpi|^2 + \|\mathbf{v}^*; H^{k+2}(B(\mathbf{0},L))\|^2 + \|w^*; H^{e_{k+1}}(B(\mathbf{0},L))\|^2 \right].
\end{aligned}
$$

Given the dependencies of K^{reg}_{k+1} and $K^{sing}_{m,k}, K^{sing}_{m,0}$, the operator D^{k+1} being arbitrary, we obtain the expected inequality for the rank $k+1$. This ends the proof.

$\square$

2.3 The case of spheres.

We end this section by computing sharp asymptotic expansions of

$$
\int_{B(\mathbf{0},L)} \gamma^3 |\nabla \varpi|^2
$$

when γ represents the distance function between one sphere of radius S and another one of radius R. Until the end of this section, we assume that, close to the origin, we have:

$$
\gamma_t(x,y) = S - \sqrt{S^2 - (x^2+y^2)}\,, \quad \gamma_b(x,y) = -R + \sqrt{R^2 - (x^2+y^2)}\,.
$$

Hence, γ depends on $r = \sqrt{x^2+y^2}$ only and we have the Taylor expansion:

$$
\left\{
\begin{array}{rcl}
\gamma_t(r) & = & \dfrac{r^2}{2S} + O(r^4)\,, \\
\gamma_b(t) & = & -\dfrac{r^2}{2R} + O(r^4)\,,
\end{array}
\right.
\qquad \gamma(r) = h + \frac{r^2}{2R_1} + \frac{r^4}{8R_3^3} + O(r^6)\,, \qquad \text{close to } r = 0, \tag{58}
$$

where R_1 and R_3 satisfy (22).
Then, we split $f = f_0 + f_1 + f_R$, where:

$$f_0 = w^*(\mathbf{0})\,, \quad f_1 = \frac{r}{2}\left(\frac{1}{R} - \frac{1}{S}\right)\mathbf{v}^*(\mathbf{0})\cdot\mathbf{e}_r + x\partial_x w^*(\mathbf{0}) + y\partial_y w^*(\mathbf{0})\,, \quad f_R = f - (f_0 + f_1).$$

Due to the linearity of the divergence problem (26)-(27), the solution ϖ admits a corresponding decomposition: $\varpi = \varpi_0 + \varpi_1 + \varpi_R$ with obvious notations. In this section, we compute separately the asymptotic expansions of the quantities

$$\int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi_0|^2, \quad \int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi_1|^2, \quad \int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi_R|^2.$$

First, as f_R corresponds to f from which the first orders in the Taylor expansion around 0 are subtracted, we may introduce $(w^*_R, \mathbf{v}^*_R) \in C^\infty(\overline{B(\mathbf{0},L)}; \mathbb{R}\times\mathbb{R}^2)$ s.t.:

$$f_R = w^*_R - \frac{1}{2}\mathrm{div}((\gamma_t + \gamma_b)\mathbf{v}^*_R)\,,$$

We have then:

$$w^*_R(\mathbf{0}) = 0\,, \quad \nabla_{x,y} w^*_R(\mathbf{0}) = \mathbf{0}\,, \quad \mathbf{v}^*_R(\mathbf{0}) = \mathbf{0}\,.$$

Hence, Proposition 9 entails that, for $h \in (0,1]$:

$$\int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi_R|^2 = O(1)\left\{\|\mathbf{u}^*; H^2(B(\mathbf{0},L))\|^2 + \|w^*; H^3(B(\mathbf{0},L))\|^2\right\}.$$

Here and in what follows, we denote by $O(1)$ a quantity which depends on (R,S,L) and h and is bounded by a constant depending only on R,S,L whatever the value of $h \in (0,1]$.

It remains to treat the two cases of ϖ_0 and ϖ_1. We remark at once that, for ϖ_0, the associated source term f_0 is constant, while, for ϖ_1, the associated source term reads $f_1 = f_1(r,\theta) = f_c r\cos(\theta) + f_s r\sin(\theta)$ where:

$$f_c = \frac{1}{2}\left(\frac{1}{R} - \frac{1}{S}\right)\mathbf{v}^*(0)\cdot\mathbf{e}_1 + \partial_x w^*(0) \qquad f_s = \frac{1}{2}\left(\frac{1}{R} - \frac{1}{S}\right)\mathbf{v}^*(0)\cdot\mathbf{e}_2 + \partial_y w^*(0).$$

We start with ϖ_0:

Proposition 15. *Under the assumption that γ is radial and satisfies* (58)*, there holds:*

$$\int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi_0|^2 = 72\pi|f_0|^2\left[\frac{R_1^2}{h} - \frac{3R_1^4}{R_3^3}|\ln(h)|\right] + O(1)\,. \tag{59}$$

Proof. By linearity, we only treat the case $f_0 = 1$. Under our symmetry assumptions, the unique solution ϖ_0 to (26)-(27) with $f = 1$ is certainly a radial function and explicit computations yield that:

$$\varpi_0(r) = \int_r^L \frac{6s}{\gamma^3(s)}\mathrm{d}s\,, \quad \partial_r\varpi_0(r) = -\frac{6r}{\gamma^3(r)}\,, \quad \forall\, r \in (0,L)\,.$$

Consequently, we have:

$$\int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi_0|^2 = 72\pi\int_0^L \frac{r^3\mathrm{d}r}{\gamma^3(r)}.$$

At this point, we note that

$$\gamma(r) = h + \frac{r^2}{2R_1} + \frac{r^4}{8R_3^3} + rem(r)$$

with $|rem(r)| \leq Cr^6$ for all $r \in (0,L)$. Consequently, introducing r_0 small enough so that we might expand γ in power series, we have:

$$\begin{aligned}\int_0^L \frac{r^3}{\gamma(r)^3}\mathrm{d}r &= \int_0^{r_0} \frac{r^3}{\gamma(r)^3}\mathrm{d}r + O(1)\\ &= \int_0^{r_0} \frac{r^3\mathrm{d}r}{(h+\frac{r^2}{2R_1})^3} - \frac{3}{8R_3^3}\int_0^{r_0} \frac{r^7\mathrm{d}r}{(h+\frac{r^2}{2R_1})^4} + O(1)\\ &= \frac{1}{h}\int_0^{\frac{r_0}{\sqrt{h}}} \frac{s^3\mathrm{d}s}{(1+\frac{s^2}{2R_1})^3} - \frac{3}{8R_3^3}\int_0^{\frac{r_0}{\sqrt{h}}} \frac{s^7\mathrm{d}s}{(1+\frac{s^2}{2R_1})^4} + O(1),\end{aligned}$$

where explicit computations yield:

$$\int_0^{\frac{r_0}{\sqrt{h}}} \frac{s^3\mathrm{d}s}{(1+\frac{s^2}{2R_1})^3} = R_1^2 + hO(1)\,, \quad \int_0^{\frac{r_0}{\sqrt{h}}} \frac{s^7\mathrm{d}s}{(1+\frac{s^2}{2R_1})^4} = 8R_1^4|\ln(h)| + O(1).$$

Finally, we obtain:

$$\int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi_0|^2 = 72\pi\left[\frac{R_1^2}{h} - \frac{3R_1^4}{R_3^3}|\ln(h)|\right] + O(1)\,.$$

$\square$

We end this section with the case of an f_1-like source term:

Proposition 16. *Assume that γ is radial and satisfies (58). Given a source term of the form $f(r,\theta) = f_c r\cos(\theta) + f_s r\sin(\theta)$ on $B(\mathbf{0},L)$, there holds:*

$$\int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi|^2 = \left(|f_c|^2+|f_s|^2\right)\frac{288\pi R_1^3}{5}|\ln(h)| + O(1)\,, \tag{60}$$

Proof. Up to shift θ with a phase and call the linearity of our problem, we prove the above result in the case $f_c = 1$ and $f_s = 0$. Then, the proof is divided into 3 steps.

Step 1: reduction to an ode. Under our symmetry assumptions, the unique solution to (26)-(27) reads $\varpi(r,\theta) = q_h(r)\cos(\theta)$ where q_h is the unique solution to the ode:

$$\frac{1}{r}\partial_r\left[\gamma(r)^3 r\partial_r q_h\right] - \frac{\gamma^3(r)q_h(r)}{r^2} = -12r\,, \qquad r\in(0,L) \tag{61}$$

$$q_h(r) = 0\,, \qquad r\in\{0,L\}\,. \tag{62}$$

The boundary condition in $r=L$ is the translation of $\varpi(r,\theta)=0$ on $r=L$ while the boundary condition in $r=0$ is derived from the condition that:

$$\pi\left[\int_0^L \gamma(r)^3\left[|\partial_r q_h(r)|^2 + \frac{|q_h(r)|^2}{r^2}\right]r\mathrm{d}r\right] = \int_{B(\mathbf{0},L)} \gamma^3|\nabla\varpi|^2\mathrm{d}x\mathrm{d}y < \infty.$$

We note that (61) rewrites:

$$\partial_{rr}q_h + \left(\frac{1}{r} + 3\frac{\gamma'(r)}{\gamma(r)}\right)\partial_r q_h - \frac{q_h}{r^2} = -\frac{12r}{\gamma^3}\,.$$

Step 2: Construction of an approximate solution. We introduce then q the unique solution to an auxiliary problem. The construction and properties of this auxiliary function are stated in the following lemma whose proof is postponed to the appendix:

Lemma 17. *Given $R_1 > 0$, there exists a unique $q \in C([0,\infty)) \cap C^\infty((0,\infty))$ solution to*

$$\partial_{ss} q + \left(\frac{1}{s} + \frac{3s}{R_1(1+\frac{s^2}{2R_1})}\right)\partial_s q - \frac{q}{s^2} = -\frac{12s}{(1+\frac{s^2}{2R_1})^3}, \qquad s \in (0,\infty),$$
$$q(0) = 0 \quad \lim_{s\to\infty} q(s) = 0. \qquad .$$

Furthermore we have the asymptotic description:

$$q(s) = \frac{48R_1^3}{5s^3} + rem(s) \quad \partial_s q(s) = -\frac{144R_1^3}{5s^4} + \frac{rem(s)}{s}. \tag{63}$$

where $|rem(s)| \le K[R_1]/s^4$ for $s > 1$.

With that auxiliary function at-hand, we set:

$$\hat{q}_h(r) = \frac{1}{h^{\frac{3}{2}}} q\left(\frac{r}{\sqrt{h}}\right) - \frac{r^2}{L^2 h^{\frac{3}{2}}} q\left(\frac{L}{\sqrt{h}}\right), \forall\, r \in (0,L), \quad \hat{\omega} = \hat{q}_h(r)\cos(\theta) \text{ on } B(\mathbf{0},L).$$

Introducing $\hat{\gamma}(r) = (h + r^2/(2R_1))$, we obtain by substitution that $\hat{q}_h$ is a solution to

$$\frac{1}{r}\partial_r\left[\hat{\gamma}(r)^3 r \partial_r \hat{q}_h\right] - \frac{\hat{\gamma}^3(r)\hat{q}_h(r)}{r^2} = -12r - \frac{1}{L^2 h^{\frac{3}{2}}} q\left(\frac{L}{\sqrt{h}}\right)\hat{\chi}(r), \qquad r \in (0,L)$$
$$\hat{q}_h(r) = 0, \qquad r \in \{0,L\}.$$

where

$$\hat{\chi}(r) = \frac{2}{r}\partial_r\left[\hat{\gamma}(r)^3 r^2\right] - \hat{\gamma}^3(r).$$

Consequently, $\hat{\omega}$ is an H^1-solution to:

$$-\frac{1}{12}\mathrm{div}[\hat{\gamma}^3 \nabla\hat{\varpi}] = \left(r + \frac{1}{12L^2 h^{\frac{3}{2}}} q\left(\frac{L}{\sqrt{h}}\right)\hat{\chi}(r)\right)\cos(\theta), \qquad \text{on } B(\mathbf{0},L),$$
$$\hat{\varpi} = 0 \qquad \text{on } r = L.$$

We note that, for $n \ge 3$, there holds:

$$\hat{I}_n := \int_{B(\mathbf{0},L)} \hat{\gamma}^n |\nabla\hat{\varpi}|^2 \mathrm{d}x\mathrm{d}y = \pi\left[\int_0^L \hat{\gamma}(r)^n \left[|\partial_r \hat{q}_h(r)|^2 + \frac{|\hat{q}_h(r)|^2}{r^2}\right] r\mathrm{d}r\right]$$

Replacing $\hat{q}_h$ with its value and noticing that $\dfrac{q(L/\sqrt{h})}{L^2 h^{\frac{3}{2}}} = O(1)$ because of (63), this entails:

$$\begin{aligned}\hat{I}_n &= \frac{\pi}{h^3}\left[\int_0^L \left(h + \frac{r^2}{2R_1}\right)^n \left[\frac{|\partial_r q(r/\sqrt{h})|^2}{h} + \frac{|q(r/\sqrt{h})|^2}{r^2}\right] r\mathrm{d}r\right] \\ &\quad + O\left(\left[1 + \int_0^L \left(h + \frac{r^2}{2R_1}\right)^{2n}\left[\frac{|\partial_r q(r/\sqrt{h})|^2}{h} + \frac{|q(r/\sqrt{h})|^2}{r^2}\right] r\mathrm{d}r\right]\right).\end{aligned}$$

We bound the first integral on the right-hand side by changing variable $r \to \sqrt{h}s$. This yields:

$$\begin{aligned}&\int_0^L \left(h + \frac{r^2}{2R_1}\right)^n \left[\frac{|\partial_r q(r/\sqrt{h})|^2}{h} + \frac{|q(r/\sqrt{h})|^2}{r^2}\right] r\mathrm{d}r \\ &\qquad = h^n \int_0^{L/\sqrt{h}} \left(1 + \frac{s^2}{2R_1}\right)^n \left[|\partial_s q(s)|^2 + \frac{|q(s)|^2}{s^2}\right] s\mathrm{d}s.\end{aligned}$$

Given the asymptotic expansion of q this entails:

$$\int_0^L \left(h + \frac{r^2}{2R_1}\right)^n \left[\frac{|\partial_r q(r/\sqrt{h})|^2}{h} + \frac{|q(r/\sqrt{h})|^2}{(r/\sqrt{h})^2}\right] r\mathrm{d}r = \begin{cases} \dfrac{h^3R_1^3}{40}\left(48^2|\ln(h)| + O(1)\right) & \text{if } n = 3 \\ h^3 O(1) & \text{if } n \geq 4. \end{cases}$$

In particular we obtain finally that:

$$\int_{B(\mathbf{0},L)} \hat{\gamma}^n |\nabla \hat{\varpi}|^2 = \begin{cases} \dfrac{288 R_1^3 \pi}{5} |\ln(h)| + O(1) & \text{for } n = 3 \\ O(1) & \text{for arbitrary } n \geq 4. \end{cases} \tag{64}$$

Step 3: Computing the distance between ϖ and $\hat{\varpi}$ We introduce the difference $D[\varpi] := \varpi - \hat{\varpi}$. It is the solution to:

$$-\frac{1}{12}\mathrm{div}[\gamma^3 \nabla D[\varpi]] = -\frac{1}{12}\mathrm{div}\left[(\hat{\gamma}^3 - \gamma^3)\nabla\hat{\varpi}\right] - \frac{1}{12L^2h^{\frac{3}{2}}} q\left(\frac{L}{\sqrt{h}}\right)\hat{\chi}(r)\cos(\theta)\,, \text{ on } B(\mathbf{0},L) \tag{65}$$

$$D[\varpi] = 0 \qquad \text{on } \partial B(\mathbf{0},L)\,. \tag{66}$$

Multypling (65) with $D[\varpi]/\gamma$, we obtain, after integration by parts:

$$\frac{1}{12}\left[\int_{B(\mathbf{0},L)} \gamma^2 |\nabla D[\varpi]|^2 - \frac{1}{4}\int_{B(\mathbf{0},L)} \nabla|\gamma|^2 \nabla |D[\varpi]|^2\right] = RHS. \tag{67}$$

On the left-hand side of this identity, we have:

$$\begin{aligned} &\frac{1}{12}\left[\int_{B(\mathbf{0},L)} \gamma^2 |\nabla D[\varpi]|^2 - \frac{1}{4}\int_{B(\mathbf{0},L)} \nabla|\gamma|^2 \nabla |D[\varpi]|^2\right] \\ &\qquad = \frac{1}{12}\left[\int_{B(\mathbf{0},L)} \gamma^2 |\nabla D[\varpi]|^2 + \frac{1}{4}\int_{B(\mathbf{0},L)} [\Delta|\gamma|^2]|D[\varpi]|^2\right] \\ &\qquad \geq \frac{1}{12}\left[\int_{B(\mathbf{0},L)} \gamma^2 |\nabla D[\varpi]|^2 + \frac{c}{2}\int_{B(\mathbf{0},L)} \gamma|D[\varpi]|^2\right]. \end{aligned}$$

as $\Delta|\gamma|^2 = 2[\gamma\Delta\gamma + |\nabla\gamma|^2] \geq 2c\gamma$ on $B(\mathbf{0},L)$. On the right-hand side, we obtain the bound above:

$$|RHS| \leq \frac{I_1}{12} + \frac{q(L/\sqrt{h})}{12L^2h^{\frac{3}{2}}} I_2$$

where, after integration by parts:

$$\begin{aligned} I_1 &= \left|\int_{B(\mathbf{0},L)} \mathrm{div}\left[(\gamma^3 - \hat{\gamma}^3)\nabla\hat{\varpi}\right]\frac{D[\varpi]}{\gamma}\right| \\ &\leq \int_{B(\mathbf{0},L)} \frac{|\gamma^3 - \hat{\gamma}^3|}{\gamma^2}|\nabla\gamma||\nabla\hat{\varpi}||D[\varpi]| + \int_{B(\mathbf{0},L)} \frac{|\gamma^3 - \hat{\gamma}^3|}{\gamma}|\nabla\hat{\varpi}||\nabla D[\varpi]| \\ &\leq \frac{C}{\varepsilon}\int_{B(\mathbf{0},L)} \frac{(\gamma^3 - \hat{\gamma}^3)^2}{\gamma^2}\left(\frac{|\nabla\gamma|^2}{|\gamma|^3} + \frac{1}{\gamma^2}\right)|\nabla\hat{\varpi}|^2 + \varepsilon\int_{B(\mathbf{0},L)} \left(\gamma^2|\nabla D[\varpi]|^2 + \gamma|D[\varpi]|^2\right). \end{aligned}$$

for arbitrary $\varepsilon > 0$. At this point, we remark that, as γ and $\hat{\gamma}$ share the same (non-vanishing) Taylor expansion up to the second order, we have, with a constant K depending only on R, S :

$$|\gamma^3 - \hat{\gamma}^3| \leq K\hat{\gamma}^4.$$

Recalling (32) to bound $\nabla\gamma$ and (64) we obtain:

$$\int_{B(\mathbf{0},L)} \frac{(\gamma^3-\hat{\gamma}^3)^2}{\gamma^2}\left(\frac{|\nabla\gamma|^2}{|\gamma|^3}+\frac{1}{\gamma^2}\right)|\nabla\hat{\varpi}|^2 \leq K\int_{B(\mathbf{0},L)} \hat{\gamma}^4|\nabla\hat{\varpi}|^2 = O(1),$$

and finally, for arbitrary positive ε, we have:

$$I_1 \leq \frac{O(1)}{\varepsilon} + \varepsilon\int_{B(\mathbf{0},L)} \left(\gamma^2|\nabla D[\varpi]|^2 + \gamma|D[\varpi]|^2\right).$$

Similarly, we bound I_2 :

$$\begin{aligned} I_2 &= \left|\int_{B(\mathbf{0},L)} \hat{\chi}(r)\cos(\theta)\frac{D[\varpi]}{\gamma}\right| \\ &\leq \frac{C}{\varepsilon}\int_0^L \left|\frac{1}{r}\partial_r\left[\hat{\gamma}(r)^3r^2\right]-\hat{\gamma}^3(r)\right|^2\frac{r\mathrm{d}r}{\gamma^3} + \varepsilon\int_{B(\mathbf{0},L)}\gamma|D[\varpi]|^2, \end{aligned}$$

where there is a constant K depending only on the characteristics of γ for which

$$\left|\frac{1}{r}\partial_r\left[\hat{\gamma}(r)^3r^2\right]-\hat{\gamma}^3(r)\right| \leq K\gamma^3$$

Hence, we have:

$$\int_0^L \left|\frac{1}{r}\partial_r\left[\hat{\gamma}(r)^3r^2\right]-\hat{\gamma}^3(r)\right|^2\frac{r\mathrm{d}r}{\gamma^3} = O(1)$$

and finally, for arbitrary $\varepsilon > 0$, there holds:

$$I_2 \leq \varepsilon\int_{B(\mathbf{0},L)}\gamma|D[\varpi]|^2 + \frac{O(1)}{\varepsilon}.$$

Introducing the above computations of the left-hand side and right-hand side into (67), noticing that

$$\frac{1}{12L^2h^{\frac{3}{2}}}q\left(\frac{L}{\sqrt{h}}\right) = O(1)$$

thanks to (63), we obtain, choosing ε sufficiently small:

$$\int_{B(\mathbf{0},L)}\gamma^2|\nabla D[\varpi]|^2 + \int_{B(\mathbf{0},L)}\gamma|D[\varpi]|^2 = O(1). \tag{68}$$

Step 4: Conclusion. We are now in position to compute the asymptotics of

$$I_h := \int_{B(\mathbf{0},L)}\gamma^3|\nabla\varpi|^2.$$

Indeed, there holds:

$$\begin{aligned} I_h &= \int_{B(\mathbf{0},L)}\hat{\gamma}^3|\nabla\hat{\varpi}|^2 + \int_{B(\mathbf{0},L)}[\gamma^3-\hat{\gamma}^3]|\nabla\hat{\varpi}|^2 \\ &\quad + \int_{B(\mathbf{0},L)}\gamma^3|\nabla D[\varpi]|^2 + 2\int_{B(\mathbf{0},L)}\gamma^3\nabla D[\varpi] : \nabla\varpi \\ &= \int_{B(\mathbf{0},L)}\hat{\gamma}^3|\nabla\hat{\varpi}|^2 + J_1 + J_2 + J_3. \end{aligned}$$

As previously, we bound $|\gamma^3-\hat{\gamma}^3| \leq K\hat{\gamma}^4$ so that recalling (64) with $n=4$ we have:

$$|J_1| \leq K\int_{B(\mathbf{0},L)}\hat{\gamma}^4|\nabla\hat{\varpi}|^2 = O(1).$$

Then, (68) implies:

$$|J_2| \leq K \int_{B(\mathbf{0},L)} \gamma^2 |\nabla D[\varpi]|^2 = O(1).$$

We conclude with similar arguments and applying Proposition 10 in case $n = 1$ (here ϖ is a solution to (26)-(27) with a special $\mathbf{v}^*$ and $w^* = 0$):

$$\begin{aligned} |J_3| &= \left| \int_{B(\mathbf{0},L)} \gamma^3 \nabla D[\varpi] : \nabla \varpi \right| \\ &\leq \int_{B(\mathbf{0},L)} \gamma^2 |\nabla D[\varpi]|^2 + \int_{B(\mathbf{0},L)} \gamma^4 |\nabla \varpi|^2 = O(1). \end{aligned}$$

Hence, there holds:

$$I_h = \int_{B(\mathbf{0},L)} \hat{\gamma}^3 |\nabla \hat{\varpi}|^2 + O(1) = \frac{288\pi R_1^3}{5} |\ln(h)| + O(1)\,,$$

because of (64). This ends the proof of Proposition 16. $\square$

3 Proof of Theorem 4

In this section, we complete the proof of Theorem 4. We first introduce some notations:

- χ_L is a truncation function which vanishes outside $B(\mathbf{0}, L)$. Namely, we set

  $$\chi_L(x,y) = \chi\left(\frac{|(x,y)|}{L}\right)\,, \quad \forall\,(x,y) \in \mathbb{R}^2\,,$$

  with $\chi \in C^\infty_c(\mathbb{R};[0,1])$ satisfying

  $$\chi(t) = 1\,,\ \forall\, t \in [-1/2, 1/2]\,, \qquad \chi(t) = 0\,,\ \forall\, t \in \mathbb{R} \setminus [-1,1]\,.$$

- we recall, that given $\ell > 0$, we denote:

  $$\mathcal{G}_\ell = \{(x,y,z) \in \mathbb{R}^3 \text{ s.t .} |(x,y)| < \ell \text{ and } z \in (\gamma_b(x,y), h + \gamma_t(x,y))\}\,,$$

- Ω^* is a smooth subdomain of $\Omega \setminus \mathcal{G}_{L/4}$ which does not depend on h and satisfies:

  $$\Omega^* \subset \mathcal{F} \text{ whatever } h \in (0,1]\,, \qquad \partial\Omega \setminus \partial\mathcal{G}_{L/2} \subset \partial\Omega^*.$$

  We note that such an Ω^* exists as, thanks to assumptions (A2) and (A3), there exists $\tilde{\delta} > 0$ depending on δ and C_{ell} such that, whatever the value of $h \in (0,1]$ there holds

  $$\{(x,y,z) \in \Omega \text{ s.t. } \mathrm{dist}((x,y,z), \partial\Omega) < \tilde{\delta}\} \setminus \mathcal{G}_{L/4} \subset \mathcal{F}\,.$$

3.1 Construction of asymptotic approximation

Let fix one boundary condition $\mathbf{u}^* \in C^\infty(\partial\Omega)$ satisfying (21). At this step, we explain how to construct a velocity-field $\mathbf{v}[\mathbf{u}^*]$ which shall approximate the solution to the Stokes problem with boundary conditions $\mathbf{u}^* \in C^\infty(\partial\Omega)$ in the regime $h << 1$.

Let split $\mathbf{u}^*$ into its tangential and normal parts (according to the tangent space in the origin):

$$\mathbf{u}^* = \mathbf{u}^*_{//} + u^*_\perp \mathbf{e}_z \in C^\infty(\partial\Omega).$$

We introduce q_{in} the unique solution to

$$\begin{aligned} -\frac{1}{12}\mathrm{div}(\gamma^3 \nabla q_{in}) &= u^*_\perp - \frac{1}{2}\mathrm{div}(\gamma_t+\gamma_b)\mathbf{u}^*_{//} - \frac{1}{2}(h-2\gamma_b)\mathrm{div}\mathbf{u}^*_{//}\,, && \text{on } B(\mathbf{0},L)\,, \\ q_{in} &= 0\,, && \text{on } \partial B(\mathbf{0},L)\,. \end{aligned} \tag{69}$$

We recall that the data γ_t, γ_b and $\mathbf{u}^*$ are smooth so that we have existence of a unique q_{in} solution to this equation satisfying:

$$q_{in} \in C^\infty(B(\mathbf{0},L)). \tag{70}$$

Second, we construct an auxiliary pair velocity-field/pressure in the aperture domain. We denote:

$$p_{in}(x,y) = \chi_L(x,y) q_{in}(x,y)\,, \quad \forall\, (x,y,z) \in \mathcal{G}_L\,. \tag{71}$$

and

$$\mathbf{v}_{in,//}(x,y,z) = \frac{1}{2}(z-(h+\gamma_t))(z-\gamma_b)\nabla_{x,y}\, p_{in} + \left(\frac{h+\gamma_t-z}{\gamma}\right)\chi_L \mathbf{u}^*_{//}\,, \tag{72}$$

$$\begin{aligned} v_{in,\perp}(x,y,z) &= \frac{1}{2}\mathrm{div}_{x,y}\left[\int_z^{h+\gamma_t}(s-(h+\gamma_t))(s-\gamma_b)\nabla_{x,y}\,p_{in}\,\mathrm{d}s\right] \\ &\quad + \int_z^{h+\gamma_t}\mathrm{div}_{x,y}\left[\left(\frac{h+\gamma_t-s}{\gamma}\right)\chi_L\mathbf{u}^*_{//}\right]\mathrm{d}s\,, \end{aligned} \tag{73}$$

for $(x,y,z) \in \mathcal{G}_L$. We note that this velocity-field vanishes outside $\mathcal{G}_L$. We keep notations to denote its trivial extension to $\mathcal{F}$ in what follows.

Combining (70) with (69)-(72)-(73), we obtain that:

(VR1) $\mathbf{v}_{in} \in C^\infty(\overline{\mathcal{F}})$ and has support in $\mathcal{G}_L$,

(VR2) div $\mathbf{v}_{in} = 0$ on $\mathcal{F}$,

(VR3) $\mathbf{v}_{in} = 0$ on $\partial\mathcal{B}$,

(VR4) $\mathbf{v}_{in} = \mathbf{u}^*$ on $\partial\Omega \cap \partial\mathcal{G}_{L/2}$

All properties satisfied by $\mathbf{v}_{in}$ are obvious but :

$$v_{in,\perp}(x,y,\gamma_b(x,y)) = u^*_\perp(x,y)\,, \quad \forall\,(x,y) \in B(\mathbf{0},L/2).$$

This property is a consequence of q_{in} solution to (69). We warn the reader that, to check this implication, one cannot commute the divergence and integral operators prior to taking the trace on $z = \gamma_b(x,y)$. Indeed, the differential operator $\mathrm{div}_{x,y}$ does not include only derivatives parallel to the boundary $z = \gamma_b(x,y)$ so that one has to expand the differential operator $\mathrm{div}_{x,y}$ inside the integral to perform this computation.

The internal velocity-field $\mathbf{v}_{in}$ does not match the boundary condition $\mathbf{u}^*$ outside $\mathcal{G}_{L/2}$ (unless $\mathbf{u}^*$ vanishes). As we expect no singular behavior of the exact solution to the Stokes problem far from the singularity, we extend the approximation in a simple way outside $\mathcal{G}_{L/2}$. Precisely, we define

$$\mathbf{v}^*_{ext} = \mathbf{u}^* - \mathbf{v}_{in}|_{\partial\Omega} \quad \text{ on } \partial\Omega.$$

As $\mathbf{v}_{in} \in C^\infty(\overline{\mathcal{F}})$ and $\mathbf{u}^*$ is smooth, we have that $\mathbf{v}^*_{ext} \in C^\infty(\partial\Omega)$. Moreover, the properties of the boundary values of $\mathbf{v}_{in}$ that we mentioned above ensure that $\mathbf{v}^*_{ext}$ vanishes on $\partial\Omega \cap \partial\mathcal{G}_{L/2}$. Consequently, we might extend $\mathbf{v}^*_{ext}$ by 0 to $\partial\Omega^*$, defining in this way a smooth function. Then, we denote $(\mathbf{v}_{ext}, q_{ext})$ the unique solution to:

$$\Delta\mathbf{v}_{ext} - \nabla q_{ext} = 0\,, \quad \text{on } \Omega^*\,, \tag{74}$$

$$\nabla\cdot\mathbf{v}_{ext} = 0\,, \quad \text{on } \Omega^*\,, \tag{75}$$

$$\mathbf{v}_{ext} = \mathbf{v}^*_{ext}\,, \quad \text{on } \partial\Omega^*\,. \tag{76}$$

This solution indeed exists as:

$$\begin{aligned}\int_{\partial\Omega^*} \mathbf{v}^*_{ext}\cdot\mathbf{n}^*\mathrm{d}\sigma &= \int_{\partial\Omega}\mathbf{v}^*_{ext}\cdot\mathbf{n}\mathrm{d}\sigma\\ &= \int_{\partial\Omega}\mathbf{u}^*\cdot\mathbf{n}\mathrm{d}\sigma - \int_{\partial\Omega}\mathbf{v}_{in}\cdot\mathbf{n}\mathrm{d}\sigma\\ &= -\int_{\partial\mathcal{F}}\mathbf{v}_{in}\cdot\mathbf{n}\mathrm{d}\sigma = 0\,.\end{aligned}$$

As $\partial\Omega^*$ and $\mathbf{v}^*_{ext}$ are smooth we also have that $\mathbf{v}_{ext}$ is smooth on $\overline{\Omega^*}$ and vanishes on $\partial\Omega^*\cap\partial[\mathcal{F}\setminus\Omega^*]$. Hence, the trivial extension of $\mathbf{v}_{ext}$ to $\mathcal{F}$ (that we still denote $\mathbf{v}_{ext}$ for simplicity) satisfies:

(VR5) $\mathbf{v}_{ext}$ is continuous piecewise smooth and has support in Ω^*

(VR6) div $\mathbf{v}_{ext} = 0$ on $\mathcal{F}$

(VR7) $\mathbf{v}_{ext} = 0$ on $\partial\mathcal{B}$,

(VR8) $\mathbf{v}_{ext} = \mathbf{u}^* - \mathbf{v}_{in}$ on $\partial\Omega$

Our asymptotic approximation $\mathbf{v}[\mathbf{u}^*]$ reads then

$$\mathbf{v}[\mathbf{u}^*] = \mathbf{v}_{in} + \mathbf{v}_{ext}. \tag{77}$$

Even though the vector-field $\mathbf{v}_{ext}$ is important in order to obtain a velocity-field $\mathbf{v}[\mathbf{u}^*]$ that matches the boundary conditions $\mathbf{u}^*$ on the whole $\partial\Omega$, this external velocity-field does not contain any information on the way the velocity-field $\mathbf{v}[\mathbf{u}^*]$ diverges when h is small. Indeed, we have the following lemma:

Lemma 18. *There exists a constant K depending on $C^{reg}_2, C_{ell}, \partial\Omega$ and δ (introduced in (A3)) such that:*

$$\int_{\mathcal{F}}|\nabla\mathbf{v}_{ext}|^2 \leq K\left[\|\mathbf{u}^*; H^2(B(\mathbf{0},L))\| + \|\mathbf{u}^*; H^{\frac12}(\partial\Omega)\|\right]^2.$$

Proof. Indeed, as solution to the Stokes problem on Ω^*, the exterior velocity-field $\mathbf{v}_{ext}$ satisfies

$$\begin{aligned}\int_{\mathcal{F}}|\nabla\mathbf{v}_{ext}|^2 &= \int_{\Omega^*}|\nabla\mathbf{v}_{ext}|^2\\ &\leq C^*\|\mathbf{v}^*_{ext}; H^{\frac12}(\partial\Omega^*)\|^2.\end{aligned}$$

where the constant C^* depends only on Ω^* which depends itself on the geometry of $\partial\Omega$ away from the singularity and δ. Moreover, recalling that $\mathbf{v}^*_{ext}$ vanishes in $\mathcal{G}_{L/2}$, we might construct a function $\zeta \in C^\infty(\overline{\Omega^*})$ such that $\zeta = 1$ on $Supp(\mathbf{v}^*_{ext})$ and $\zeta = 0$ on $\partial\Omega^*\setminus\partial\Omega$. We have then, introducing $\Gamma[\mathbf{u}^*] \in H^1(\Omega)$ a lifting of $\mathbf{u}^*$ on the whole Ω:

$$\begin{aligned}\|\mathbf{v}^*_{ext}; H^{\frac12}(\partial\Omega^*)\| &\leq \|\zeta\mathbf{u}^*; H^{\frac12}(\partial\Omega^*)\| + \|\zeta\mathbf{v}_{in}; H^{\frac12}(\partial\Omega^*)\|\\ &\leq C^*\|\zeta\Gamma[\mathbf{u}^*]; H^1(\Omega^*)\| + \|\zeta\mathbf{v}_{in}; H^1(\Omega^*)\|\\ &\leq C^*_\zeta\left(\|\mathbf{u}^*; H^{\frac12}(\partial\Omega)\| + \|\mathbf{v}_{in}; H^1(\Omega^*)\|\right).\end{aligned}$$

As Ω^* remains away from the singularity, we might bound from below γ by a constant $\hat{\delta}$ depending only on C_{ell} on $\mathcal{G}_L\cap\Omega^*$ and apply the explicit formulas for $\mathbf{v}_{in}$ to obtain the following bounds:

$$\|\mathbf{v}_{in}; H^1(\Omega^*)\| \leq C^*\left(1+\frac{1}{\hat{\delta}^3}\right)\left(\|\gamma_b; C^2(\overline{B(\mathbf{0},L)})\| + \|\gamma_t; C^2(\overline{B(\mathbf{0},L)})\|\right)\times\ldots$$
$$\ldots\times\left(\|q_{in}; H^3(B(\mathbf{0},L)\setminus B(\mathbf{0},L/2))\| + \|\mathbf{u}^*; H^2(B(\mathbf{0},L))\|\right).$$

We now wish to apply Proposition 11 to q_{in} in order to yield a constant that bounds its H^3-norm by something which does not depend on h. However, one term in the right-hand side of (69) depends on h. To avoid this difficulty we expand $q_{in} = q_{in}^{(1)} + hq_{in}^{(2)}$ with pressure field components defined as respective solutions to:

$$\begin{aligned} -\frac{1}{12}\mathrm{div}(\gamma^3 \nabla q_{in}^{(1)}) &= (u_\perp^* + \gamma_b \mathrm{div}\mathbf{u}_{/\!/}^*) - \frac{1}{2}\mathrm{div}(\gamma_t + \gamma_b)\mathbf{u}_{/\!/}^* \,, && \text{on } B(\mathbf{0}, L)\,, \\ q_{in}^{(1)} &= 0\,, && \text{on } \partial B(\mathbf{0}, L)\,. \end{aligned} \tag{78}$$

and

$$\begin{aligned} -\frac{1}{12}\mathrm{div}(\gamma^3 \nabla q_{in}^{(2)}) &= -\frac{1}{2}\mathrm{div}\mathbf{u}_{/\!/}^* \,, && \text{on } B(\mathbf{0}, L)\,, \\ q_{in}^{(2)} &= 0\,, && \text{on } \partial B(\mathbf{0}, L)\,. \end{aligned} \tag{79}$$

In particular, we remark that $q_{in}^{(1)}$ is a solution to (26)-(27) with a source term f given by (28) associated with

$$w_\perp^* = u_\perp^* + \gamma_b \mathrm{div}\mathbf{u}_{/\!/}^* \qquad \mathbf{v}^* = \mathbf{u}_{/\!/}^* \,,$$

and that $q_{in}^{(2)}$is a solution to (26)-(27) with a source term f given by (28) associated with

$$w_\perp^* = -\frac{1}{2}\mathrm{div}\mathbf{u}_{/\!/}^* \qquad \mathbf{v}^* = 0.$$

Hence, we apply Proposition 11 to both $q_{in}^{(1)}$ and $q_{in}^{(2)}$ and obtain:

$$\|q_{in}^{(1)}; H^3(B(\mathbf{0}, L) \setminus B(\mathbf{0}, L/2))\| + h\|q_{in}^{(2)}; H^3(B(\mathbf{0}, L) \setminus B(\mathbf{0}, L/2))\| \leq K_{reg}^1 \|\mathbf{u}^*; H^2(B(\mathbf{0}, L))\|.$$

This entails finally:

$$\begin{aligned} \|\mathbf{v}_{in}; H^1(\Omega^*)\| \leq C^* \left(1 + \frac{1}{\hat{\delta}^3}\right) \left(\|\gamma_b; C^2(\overline{B(\mathbf{0}, L)})\| + \|\gamma_t; C^2(\overline{B(\mathbf{0}, L)})\|\right) \times \dots \\ \dots \times K_1^{reg} \|\mathbf{u}^*; H^2(B(\mathbf{0}, L))\|. \end{aligned} \tag{80}$$

□

3.2 Main steps of the proof of Theorem 4

We fix now $\mathbf{u}^* \in C^\infty(\partial\Omega)$ satisfying (21) as in the assumptions of Theorem 4. We split this boundary condition in

$$\mathbf{u}^* = \mathbf{u}_{as}^* + \mathbf{u}_R^*$$

with:

$$\begin{aligned} \mathbf{u}_{as}^* &= u_\perp^*(\mathbf{0})\mathbf{e}_z + \mathbf{u}_{/\!/}^*(\mathbf{0}) + (x\partial_x u_\perp^*(\mathbf{0}) + y\partial_y u_\perp^*(\mathbf{0}))\mathbf{e}_z \,, \\ \mathbf{u}_R^* &= \mathbf{u}^* - \mathbf{u}_{as}^* \,. \end{aligned}$$

Note that, straightforward computations entail:

$$\int_{\partial\Omega} \mathbf{u}_i^* \cdot \mathbf{n} \mathrm{d}\sigma = 0, \qquad \mathbf{u}_i^* \in C^\infty(\partial\Omega), \quad \forall\, i \in \{as, R\}\,.$$

So, we might define the solutions of Stokes system (1)–(4)(+(6)) as given by Theorem 3 for boundary data $\mathbf{u}^*, \mathbf{u}_{as}^*, \mathbf{u}_R^*$. We denote these solutions $(\mathbf{u}, p)$, $(\mathbf{u}_{as}, p_{as})$, $(\mathbf{u}_R, p_R)$ respectively. We also introduce the asymptotic approximations:

$$\mathbf{v}_{as} = \mathbf{v}[\mathbf{u}_{as}^*] \quad \mathbf{v}_R = \mathbf{v}[\mathbf{u}_R^*].$$

Let denote $\mathbf{v} = \mathbf{v}_{as}$. Then, as $\mathbf{u}^*_{as,//}$ is constant, we have that

$$(h - 2\gamma_b)\mathrm{div}\mathbf{u}^*_{as,//} = 0$$

so that the pressure associated with $\mathbf{v}_{as}$ satisfies (18)-(19) and, consequently, $\mathbf{v}_{as}$ satisfies (16)-(17) in $\mathcal{G}_{L/2}$. To complete the proof, it remains to compute

$$\|\mathbf{u} - \mathbf{v}; V\| = \|\mathbf{u}_{as} + \mathbf{u}_R - \mathbf{v}_{as}; V\| \leq \|\mathbf{v}_{as} - \mathbf{u}_{as}; V\| + \|\mathbf{u}_R; V\|.$$

Next subsection is devoted to the computation of $\|\mathbf{u}_R; V\|$ and the following one to $\|\mathbf{v}_{as} - \mathbf{u}_{as}; V\|$. In particular, the proof of Theorem 4 is completed by applying Proposition 19 and Proposition 21 in the general case and by applying Proposition 19, Proposition 22 and Proposition 23 in the radial case.

3.3 Asymptotics of $\mathbf{u}_R$

We start with the remainder term $\mathbf{u}_R$. Keeping notation $\mathbf{v}_R$ for $\mathbf{v}[\mathbf{u}^*_R]$, we prove:

Proposition 19. *There exists a constant K depending on $\partial\Omega$, δ, C_3^{reg}, C_{cvx}, C_{ell} and L such that, if $h \in (0,1]$, there holds:*

$$\int_{\mathcal{F}} |\nabla \mathbf{u}_R|^2 \leq K \left[\|\mathbf{u}^*; H^3(B(\mathbf{0}, L))\|^2 + \|\mathbf{u}^*; H^{\frac{1}{2}}(\partial\Omega)\|^2 \right].$$

We recall that thanks to the variational characterization Proposition 6, $\mathbf{u}_R$ realizes the minimum of V norms among divergence-free velocity-fields $\mathbf{w}$ satisfying the same boundary conditions as $\mathbf{u}_R$ (*i.e.* $\mathbf{w} \in V[\mathbf{u}^*_R]$ with our notations). By construction, we have $\mathbf{v}_R \in V[\mathbf{u}^*_R]$ so that the above proposition is a consequence of the lemma:

Lemma 20. *There exists a constant K depending on on $\partial\Omega$, δ, C_3^{reg}, C_{cvx}, C_{ell} and L for which, if $h \in (0,1]$:*

$$\int_{\mathcal{F}} |\nabla \mathbf{v}_R|^2 \leq K \left[\|\mathbf{u}^*; H^3(B(\mathbf{0}, L))\|^2 + \|\mathbf{u}^*; H^{\frac{1}{2}}(\partial\Omega)\|^2 \right]. \tag{81}$$

Proof. We drop index R in $\mathbf{v}$ for the whole proof. We recall that, by construction, we have $\mathbf{v} = \mathbf{v}_{in} + \mathbf{v}_{ext}$ where $\mathbf{v}_{in}$ and $\mathbf{v}_{ext}$ are computed *via* (72)-(73) and (74)-(76) respectively. We first apply Lemma 18 showing that we only have to focus on the following contribution of $\mathbf{v}_{in}$:

$$\int_{\mathcal{F}} |\nabla \mathbf{v}_{in}|^2 = \int_{\mathcal{G}_L} |\nabla \mathbf{v}_{in}|^2 .$$

Explicit computations show that:

$$\begin{aligned}
|\nabla_{x,y}\mathbf{v}_{in,//}| &\leq C_L \sum_{i=t,b} \left(|\nabla_{x,y}\gamma_i|\gamma|\nabla_{x,y}p_{in}| + \gamma^2|\nabla^2_{x,y}p_{in}| + \left(\frac{|\nabla_{x,y}\gamma_i|}{\gamma} + 1\right)|\mathbf{u}^*_{R,//}| + |\nabla_{x,y}\mathbf{u}^*_{R,//}| \right), \\
|\partial_z \mathbf{v}_{in,//}| &\leq C_L \left(\gamma|\nabla_{x,y}p_{in}| + \frac{1}{\gamma}|\mathbf{u}^*_{R,//}| \right), \\
|\partial_z v_{in,\perp}| &\leq C_L |\nabla_{x,y}\mathbf{v}_{in,//}|,
\end{aligned}$$

and:

$$\begin{aligned}
|\nabla_{x,y}v_{in,\perp}| \leq C_L \Bigg\{ &\sum_{i=t,b} \left((|\nabla_{x,y}\gamma_i|^2\gamma + |\nabla^2_{x,y}\gamma_i|\gamma^2)|\nabla_{x,y}p_{in}| + \gamma^2|\nabla_{x,y}\gamma_i||\nabla^2_{x,y}p_{in}| \right) \\
&+ \left(|\nabla^2_{x,y}\gamma_t| + |\nabla^2_{x,y}\gamma| + |\nabla_{x,y}\gamma_t| + |\nabla_{x,y}\gamma| + \frac{|\nabla_{x,y}\gamma_t||\nabla_{x,y}\gamma| + |\nabla_{x,y}\gamma|^2}{\gamma} + \gamma \right)|\mathbf{u}^*_{R,//}| \\
&+ (|\nabla_{x,y}\gamma_t| + |\nabla\gamma| + \gamma)|\nabla_{x,y}\mathbf{u}^*_{R,//}| + \gamma|\nabla^2_{x,y}\mathbf{u}^*_{R,//}| + \gamma^3|\nabla^3_{x,y}p_{in}| \Bigg\},
\end{aligned}$$

with C_L a constant depending on L. Introducing that, for all $(x,y) \in B(\mathbf{0},L)$, there holds

$$|\nabla_{x,y}\gamma_b| + |\nabla_{x,y}\gamma_t| + |\nabla_{x,y}\gamma| \leq K[C_2^{reg}, C_{ell}]\gamma^{\frac{1}{2}}\,, \quad |\nabla^2_{x,y}\gamma_i| + |\nabla^2_{x,y}\gamma| + |\nabla^2_{x,y}\gamma| \leq K[C_2^{reg}]\,,$$

we obtain that, on $\mathcal{G}_L$, there holds:

$$|\nabla \mathbf{v}_{in}| \leq K\left(\gamma|\nabla_{x,y}p_{in}| + \gamma^2|\nabla^2_{x,y}p_{in}| + \gamma^3|\nabla^3_{x,y}p_{in}| + \left(1+\frac{1}{\gamma}\right)|\mathbf{u}^*_{R,//}| + |\nabla_{x,y}\mathbf{u}^*_{R,//}| + \gamma|\nabla^2_{x,y}\mathbf{u}^*_{R,//}|\right).$$

Integrating this inequality entails that:

$$\begin{aligned}\int_{\mathcal{G}_L} |\nabla \mathbf{v}_{in}|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z &\leq K[C_{ell}, C_2^{reg}, L]\|\mathbf{u}^*_{R,//}; H^2(B(\mathbf{0},L))\|^2 \\ &+ K[C_{ell}, C_2^{reg}, L]\int_{B(\mathbf{0},L)}\left(\gamma^3|\nabla_{x,y}p_{in}|^2 + \gamma^5|\nabla^2_{x,y}p_{in}|^2 + \gamma^7|\nabla^3_{x,y}p_{in}|^2 + \frac{|\mathbf{u}^*_{R,//}|^2}{\gamma}\right)\mathrm{d}x\mathrm{d}y. \quad (82)\end{aligned}$$

In this last identity, we note that, for $k \in \{1,2,3\}$:

$$|\nabla^k_{x,y}p_{in}(x,y)| \leq |\nabla^k_{x,y}q_{in}(x,y)| + C_L\sum_{j=0}^{k-1}\mathbf{1}_{B(\mathbf{0},L)\setminus B(\mathbf{0},L/2)}(x,y)|\nabla^j_{x,y}q_{in}(x,y)|\,, \quad \forall\,(x,y) \in B(\mathbf{0},L).$$

So, we split again $q_{in} = q^{(1)}_{in} + hq^{(2)}_{in}$ with $q^{(1)}_{in}$ and $q^{(2)}_{in}$ defined respectively as the solutions to

$$\begin{aligned} -\frac{1}{12}\mathrm{div}_{x,y}(\gamma^3\nabla_{x,y}q^{(1)}_{in}) &= (u^*_{R,\perp} + \gamma_b \mathrm{div}_{x,y}\mathbf{u}^*_{R,//}) - \frac{1}{2}\mathrm{div}_{x,y}(\gamma_t + \gamma_b)\mathbf{u}^*_{R,//}\,, && \text{on } B(\mathbf{0},L)\,, \\ q^{(1)}_{in} &= 0\,, && \text{on } \partial B(\mathbf{0},L)\,. \end{aligned} \quad (83)$$

and

$$\begin{aligned} -\frac{1}{12}\mathrm{div}_{x,y}(\gamma^3\nabla_{x,y}q^{(2)}_{in}) &= -\frac{1}{2}\mathrm{div}_{x,y}\mathbf{u}^*_{R,//}\,, && \text{on } B(\mathbf{0},L)\,, \\ q^{(2)}_{in} &= 0\,, && \text{on } \partial B(\mathbf{0},L)\,. \end{aligned} \quad (84)$$

So we have that $q^{(2)}_{in}$ is a solution to (26)-(27) with a source term f given by (28) associated with

$$w^*_\perp = -\frac{1}{2}\mathrm{div}_{x,y}\mathbf{u}^*_{R,//} \qquad \mathbf{v}^* = 0.$$

where we note that $\|\mathrm{div}_{x,y}\mathbf{u}^*_{R,//}; L^\infty(B(0,L))\| \leq C\|\mathbf{u}^*_R; H^3(B(\mathbf{0},L))\|$. Applying propositions 11, 12 and 10 to $q^{(2)}_{in}$, we obtain that,

$$\begin{aligned}\int_{B(\mathbf{0},L)}\left(\gamma^3|\nabla_{x,y}q^{(2)}_{in}|^2 + \gamma^5|\nabla^2_{x,y}q^{(2)}_{in}|^2 + \gamma^7|\nabla^3_{x,y}q^{(2)}_{in}|^2\right)\mathrm{d}x\mathrm{d}y \\ + \|q^{(2)}_{in}; H^3(B(\mathbf{0},L)\setminus B(\mathbf{0},L/2))\| \leq \frac{K\|\mathbf{u}^*_R; H^3(B(\mathbf{0},L))\|^2}{h}.\end{aligned}$$

with K depending on C_3^{reg}, C_{cvx}, C_{ell} and L.

As for $q^{(1)}_{in}$, we remark that it is a solution to (26)-(27) with a source term f given by (28) associated with

$$w^*_\perp = u^*_{R,\perp} + \gamma_b \mathrm{div}_{x,y}\mathbf{u}^*_{R,//} \qquad \mathbf{v}^* = \mathbf{u}^*_{R,//}\,.$$

In this case, we have that $w^*_\perp(\mathbf{0}) = 0$ and also that $\nabla w^*_\perp(\mathbf{0}), \mathbf{v}^*(\mathbf{0})$ vanish. Consequently, we are in position to apply propositions 11, 12 and 9 to $q^{(1)}_{in}$, yielding that:

$$\begin{aligned}\int_{B(\mathbf{0},L)}\left(\gamma^3|\nabla_{x,y}q^{(1)}_{in}|^2 + \gamma^5|\nabla^2_{x,y}q^{(1)}_{in}|^2 + \gamma^7|\nabla^3_{x,y}q^{(1)}_{in}|^2\right) \\ + \|q^{(1)}_{in}; H^3(B(\mathbf{0},L)\setminus B(\mathbf{0},L/2))\| \leq K\|\mathbf{u}^*_R; H^3(B(\mathbf{0},L))\|^2\,,\end{aligned}$$

with K depending again on C_3^{reg}, C_{cvx}, C_{ell} and L. Combining the computations for $q_{in}^{(1)}$ and $q_{in}^{(2)}$, and arguing that

$$\|\mathbf{u}_R^*; H^3(B(\mathbf{0},L))\| \leq C\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|$$

we get finally:

$$\int_{B(\mathbf{0},L)} \left(\gamma^3|\nabla_{x,y}p_{in}|^2 + \gamma^5|\nabla^2_{x,y}p_{in}|^2 + \gamma^7|\nabla^3_{x,y}p_{in}|^2\right) \leq K\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2\,, \tag{85}$$

with K depending on C_3^{reg}, C_{cvx}, C_{ell} and L.

For the last term on the right-hand side of (82), we add that $\mathbf{u}^*_{//}(\mathbf{0}) = \mathbf{0}$, implying:

$$|\mathbf{u}^*_{R,//}(x,y)| \leq (|x|+|y|)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|\,, \qquad \text{on } B(\mathbf{0},L)\,.$$

Consequently, going to polar coordinates yields (recall γ satisfies (31) whatever the value of $h \in (0,1]$):

$$\int_{B(\mathbf{0},L)} \frac{|\mathbf{u}^*_{R,//}|^2}{\gamma} \leq \frac{2\pi\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2}{C_{ell}} \int_0^L \frac{r^3 \mathrm{d}r}{r^2} \leq \frac{\pi L^2}{C_{ell}}\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^3.$$

This ends the proof. $\square$

3.4 Asymptotics of $\mathbf{v}_{as} - \mathbf{u}_{as}$

We proceed with a first bound on the singular term in the general case (*i.e.*, without structure assumption on γ). We prove:

Proposition 21. *If $h \in (0,1]$, there exists a constant K depending on $\partial\Omega$, δ, C_3^{reg}, C_{cvx}, C_{ell} and L such that there holds:*

$$\int_{\mathcal{F}} |\nabla \mathbf{u}_{as} - \nabla \mathbf{v}_{as}|^2 \leq K[|u^*_\perp(\mathbf{0})|^2|\ln(h)| + \|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2].$$

Proof. We recall that $\mathbf{v}_{as} = \mathbf{v}_{in} + \mathbf{v}_{ext}$. It is sufficient to compute a constant K (independent of h) such that, if $h \in (0,1]$, for all $\mathbf{w} \in \mathcal{V}_0$, we have:

$$\left|\int_{\mathcal{F}} (\Delta \mathbf{v}_{in} - \nabla p_{in}) \cdot \mathbf{w}\right| \leq K\left[\int_{\mathcal{F}} |\nabla \mathbf{w}|^2\right]^{\frac{1}{2}}.$$

(with p_{in} given by (71)). Indeed, we have then:

$$|\langle \Delta \mathbf{v}_{as} - \nabla p_{in}, \mathbf{w}\rangle| \leq \left|\int_{\mathcal{F}} (\Delta \mathbf{v}_{in} - \nabla p_{in}) \cdot \mathbf{w}\right| + \left|\int_{\mathcal{F}} \nabla \mathbf{v}_{ext} : \nabla \mathbf{w}\right|$$

Applying Lemma 18 together with the remark that the mapping $\mathbf{u}^* \mapsto \mathbf{u}^*_{as}$ is linear continuous $H^3(B(\mathbf{0},L)) \to H^2(B(\mathbf{0},L)) \cap H^{\frac{1}{2}}(\partial\Omega)$, we bound the other term by:

$$\left|\int_{\mathcal{F}} \nabla \mathbf{v}_{ext} : \nabla \mathbf{w}\right| \leq K\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|\|\mathbf{w}; V\|\,.$$

with K depending on $\partial\Omega$, C_2^{reg}, C_{ell} and δ.

We emphasize that $\mathbf{v}_{in}$ and p_{in} have support in $\mathcal{G}_L$ so that:

$$I[\mathbf{w}] := \int_{\mathcal{F}} (\Delta \mathbf{v}_{in} - \nabla p_{in}) \cdot \mathbf{w} = \int_{\mathcal{G}_L} (\Delta \mathbf{v}_{in} - \nabla p_{in}) \cdot \mathbf{w}\,.$$

By construction, we have that:

$$\partial_{zz}\mathbf{v}_{in,//} = \nabla_{x,y}p_{in}\,, \qquad \partial_z p_{in} = 0$$

Consequently, there holds:

$$\Delta \mathbf{v}_{in} - \nabla p_{in} = \Delta_{x,y} \mathbf{v}_{in,//} + \Delta v_{in,\perp} \mathbf{e}_z.$$

For any $\mathbf{w} \in \mathcal{V}_0$ we can then bound $I[\mathbf{w}]$ by integrating by parts:

$$\begin{aligned} |I[\mathbf{w}]| &= \left| \int_{\mathcal{G}_L} \nabla_{x,y} \mathbf{v}_{in,//} : \nabla_{x,y} \mathbf{w} + \int_{\mathcal{G}_L} \nabla v_{in,\perp} \cdot \nabla w_\perp \right| \\ &\leq \left[\int_{\mathcal{G}_L} |\nabla_{x,y} \mathbf{v}_{in,//}|^2 + |\nabla v_{in,\perp}|^2 \right]^{\frac{1}{2}} \|\nabla \mathbf{w}; L^2(\mathcal{F})\|. \end{aligned}$$

The remainder of the proof follows the line of the proof of Lemma 20. First, we bound $|\nabla_{x,y} \mathbf{v}_{in,//}|$ and $|\nabla v_{in,\perp}|$ as in the proof of this lemma, this yields:

$$\begin{aligned} &|\nabla_{x,y} \mathbf{v}_{in,//}| + |\nabla v_{in,\perp}| \\ &\qquad \leq K[C_{ell}, C_2^{reg}] \Bigg(\gamma^{\frac{3}{2}} |\nabla_{x,y} p_{in}| + \gamma^2 |\nabla^2_{x,y} p_{in}| + \gamma^3 |\nabla^3 p_{in}| \\ &\qquad\qquad + \left(1 + \frac{1}{\gamma^{\frac{1}{2}}}\right) |\mathbf{u}^*_{as,//}| + |\nabla_{x,y} \mathbf{u}^*_{as,//}| + |\nabla^2_{x,y} \mathbf{u}^*_{as,//}| \Bigg). \end{aligned}$$

We obtain then:

$$\begin{aligned} &\left| \int_{\mathcal{G}_L} |\nabla_{x,y} \mathbf{v}_{in,//}|^2 + |\nabla v_{in,\perp}|^2 \right| \\ &\qquad \leq C \left[\|\mathbf{u}^*_{as,//}; H^2(B(\mathbf{0}, L))\|^2 + \int_{B(\mathbf{0},L)} \left(\gamma^4 |\nabla_{x,y} p_{in}|^2 + \gamma^5 |\nabla^2_{x,y} p_{in}|^2 + \gamma^7 |\nabla^3_{x,y} p_{in}|^2 \right) \right], \quad (86) \end{aligned}$$

and we bound the last integrals on the right-hand side by computing p_{in} with respect to q_{in} and applying Propositions 10, 11 and 12 as in the proof of Lemma 20. Note that q_{in} has only one component because $\mathrm{div}_{x,y} \mathbf{u}^*_{as,//} = 0$. $\square$

We end this section by considering the case where the aperture admits a cylindrical invariance: $\gamma = \gamma(\sqrt{x^2 + y^2})$. In this case we introduce $(\mathbf{u}_0, p_0)$ and $(\mathbf{u}_1, p_1)$ the solutions to the Stokes system associated with boundary conditions:

$$\begin{aligned} \mathbf{u}^*_0 &= u^*_\perp(\mathbf{0}) \mathbf{e}_z \,, \\ \mathbf{u}^*_1 &= \mathbf{u}^*_{//}(\mathbf{0}) + (x \partial_x u^*_\perp(\mathbf{0}) + y \partial_y u^*_\perp(\mathbf{0})) \mathbf{e}_z \,, \end{aligned}$$

We note that straightforward computations entail:

$$\int_{\partial\Omega} \mathbf{u}^*_i \cdot \mathbf{n} \mathrm{d}\sigma = 0, \qquad \mathbf{u}^*_i \in C^\infty(\partial\Omega), \quad \forall\, i \in \{0, 1\}\,.$$

So, we have indeed existence and uniqueness of the pairs $(\mathbf{u}_i, p_i)_{i=0,1}$. We also introduce $\mathbf{v}_0 = \mathbf{v}[\mathbf{u}^*_0]$, $\mathbf{v}_1 = \mathbf{v}[\mathbf{u}^*_1]$ the respective approximations of $\mathbf{u}_0$ and $\mathbf{u}_1$ constructed applying the steps depicted in Section 3.1. We note that, due to the linearity of the Stokes problem and of our construction, we have:

$$\mathbf{u}_{as} = \mathbf{u}_0 + \mathbf{u}_1 \,, \quad \mathbf{v}_{as} = \mathbf{v}_0 + \mathbf{v}_1 \,.$$

First, remarking that $\mathbf{u}^*_1(\mathbf{0}) \cdot \mathbf{e}_z$ vanishes (so that there is no logarithmic terms yielding from the application of Proposition 10), we reproduce the computations in the proof of the previous proposition and obtain at first:

Proposition 22. *If $h \in (0,1]$, there exists a constant K depending on $\partial\Omega$, δ, C_3^{reg}, C_{cvx}, C_{ell} and L such that there holds:*

$$\int_{\mathcal{F}} |\nabla \mathbf{u}_1 - \nabla \mathbf{v}_1|^2 \leq K \|\mathbf{u}^*; H^3(B(\mathbf{0}, L))\|^2.$$

We complete the study of $\mathbf{u}_{as} - \mathbf{v}_{as}$ by computing the asymptotics of the most singular term: $\mathbf{u}_0 - \mathbf{v}_0$. We prove:

Proposition 23. *If $h \in (0,1]$ and γ is radial, there exists a constant K depending on $\partial\Omega$, δ, C_3^{reg}, C_{cvx}, C_{ell} and L such that there holds:*

$$\int_{\mathcal{F}} |\nabla \mathbf{u}_0 - \nabla \mathbf{v}_0|^2 \leq K \|\mathbf{u}^*; H^2(B(\mathbf{0}, L))\|^2.$$

Proof. As the mapping $\mathbf{u}^* \mapsto \mathbf{u}_0^*$ is continuous $H^2(B(\mathbf{0}, L)) \to H^3(B(\mathbf{0}, L)) \cap H^{\frac{1}{2}}(\partial\Omega)$, we treat only the case $\mathbf{u}_0^* = \mathbf{e}_z$ and we drop superfluous index 0.

In this cylindrical case, a particular feature of $\mathbf{v}_0$ is that, in the problem solved by q_{in} the weight γ is invariant by rotation around the origin and the source term is a constant. Consequently, q_{in} is a radial function and an explicit formula is available (as in the proof of Proposition 15). Up to assume that χ_L is also radial, we get that $\mathbf{v}_{in,//}$ is directed along the radial unit vector $\mathbf{e}_r$. More precisely, we have:

$$\mathbf{v}_{in,//}(x,y,z) = v_r(r,z)\mathbf{e}_r\,, \qquad \text{a.e. in } \mathcal{G}_L. \tag{87}$$

with an explicit formula in the aperture:

$$v_r(r,z) = -\frac{3r}{\gamma^3(r)}(z - (h + \gamma_t(r)))(z - \gamma_b(r))\,, \qquad \text{a.e. in } \mathcal{G}_{L/2}.$$

Following the proof of Proposition 21 in the previous section, to bound the distance between $\mathbf{v}_0$ and $\mathbf{u}_0$, we compute integrals

$$I_{in}[\mathbf{w}] := \int_{\mathcal{G}_L} (\Delta \mathbf{v}_{in} - \nabla p_{in}) \cdot \mathbf{w}.$$

Again, as in the previous section, explicit computations and integrations by parts yield that:

$$I_{in}[\mathbf{w}] = \int_{\mathcal{G}_L} \Delta_{x,y} \mathbf{v}_{in,//} \cdot \mathbf{w}_{//} - \int_{\mathcal{G}_L} \nabla_{x,y} v_{in,\perp} \cdot \nabla_{x,y} w_\perp - \int_{\mathcal{G}_L} \partial_z v_{in,\perp} \partial_z w_\perp.$$

Here, we introduce that

$$\partial_z w_\perp = -\mathrm{div}_{x,y} \mathbf{w}_{//} \qquad \partial_z v_{in,\perp} = -\mathrm{div}_{x,y} \mathbf{v}_{in,//}\,. \tag{88}$$

We plug these identities in the above integrals and integrate by parts. Because of the radial form of $\mathbf{v}_{in,//}$ (see (87)), we have

$$\nabla_{x,y} \mathrm{div}_{x,y} \mathbf{v}_{in,//} = \Delta_{x,y} \mathbf{v}_{in,//}\,.$$

This entails:

$$I_{in}[\mathbf{w}] = \int_{\mathcal{G}_L} 2\Delta_{x,y} \mathbf{v}_{in,//} \cdot \mathbf{w}_{//} - \int_{\mathcal{G}_L} \nabla_{x,y} v_{in,\perp} \cdot \nabla_{x,y} w_\perp$$

For the last integral, we bound $|\nabla_{x,y} v_{in,\perp}|$ similarly as in the proof of Lemma 20. Introducing the bounds on γ_b, γ_t and γ, this entails:

$$|\nabla_{x,y} v_{in,\perp}| \leq C \left(\gamma^2 |\nabla_{x,y} p_{in}| + \gamma^{\frac{5}{2}} |\nabla^2_{x,y} p_{in}| + \gamma^3 |\nabla^3_{x,y} p_{in}| \right)$$

so that, introducing now that $p_{in} = \chi_L q_{in}$ and applying Proposition 11:

$$\int_{\mathcal{G}_L} |\nabla_{x,y} v_{in,\perp}|^2 \leq K[C_2^{reg}, C_{ell}, C_{cvx}] \left(\int_{B(\mathbf{0},L)} \left[\gamma^5 |\nabla q_{in}|^2 + \gamma^6 |\nabla^2 q_{in}|^2 + \gamma^7 |\nabla^3 q_{in}|^2 \right] + 1 \right).$$

Applying Propositions 10 and 12 to q_{in} entails:

$$\int_{\mathcal{G}_L} |\nabla_{x,y} v_{in,\perp}|^2 \leq K[C_3^{reg}, C_{ell}, C_{cvx}] \,.$$

Then, we truncate $\mathbf{w}_{//}$ with $\chi_{L/2}$ that vanish on the lateral boundaries of $\partial\mathcal{G}_{L/2}$ (and is equal to 1 on $\mathcal{G}_{L/4}$) and we obtain that:

$$\left| \int_{\mathcal{G}_L} \Delta_{x,y} \mathbf{v}_{in,//} \cdot \mathbf{w}_{//} \right| \leq \int_{\mathcal{G}_{L/2}} |\Delta_{x,y} \mathbf{v}_{in,//}| \cdot |\chi_{L/2} \mathbf{w}_{//}| + C_L \|\mathbf{v}_{in,//} \,;\, H^1(\Omega \setminus \mathcal{G}_{L/4})\| \|\mathbf{w}_{//} \,;\, H^1(\Omega)\|.$$

With similar arguments as in the proof of Lemma 18 (see (80)), we bound the last term on the right-hand side:

$$\|\mathbf{v}_{in,//} \,;\, H^1(\Omega \setminus \mathcal{G}_{L/4})\| \leq K[C_{ell}, C_2^{reg}, \delta].$$

Finally, we apply the formula for $\mathbf{v}_{in,//}$ which implies in particular that:

$$\Delta_{x,y} \mathbf{v}_{in,//} = \left[r\partial_{rr} \left(\frac{v_r}{r} \right) + 3\partial_r \left(\frac{v_r}{r} \right) \right] .$$

Hence introducing that $\mathbf{w}$ vanishes on the upper and lower boundaries of $\mathcal{G}_{L/2}$, we bound with Cauchy Schwartz inequalities and a Hardy inequality in the z direction:

$$\begin{aligned}
\int_{\mathcal{G}_L} |\Delta_{x,y} \mathbf{v}_{in,//}| \cdot |\chi_{L/2} \mathbf{w}_{//}| &\leq \int_0^{2\pi} \int_0^L \int_{\gamma_b(r)}^{h+\gamma_t(r)} (z - \gamma_b) \, |\Delta_{x,y} \mathbf{v}_{in,//}| \cdot \frac{|\chi_{L/2} \mathbf{w}_{//}|}{z - \gamma_b} \mathrm{d}r\mathrm{d}z\mathrm{d}\theta \\
&\leq C \left[\int_{\mathcal{G}_L} \left| \gamma \left[r\partial_{rr} \left(\frac{v_r}{r} \right) + 3\partial_r \left(\frac{v_r}{r} \right) \right] \right|^2 \right]^{\frac{1}{2}} \|\nabla \mathbf{w}; L^2(\mathcal{F})\|.
\end{aligned}$$

We emphasize that the constant C is universal and in particular independent of h. With the explicit formula for v_r we get:

$$\left| \gamma \left[r\partial_{rr} \left(\frac{v_r}{r} \right) + 3\partial_r \left(\frac{v_r}{r} \right) \right] \right| \leq K[C_2^{reg}, C_{ell}] \frac{r}{\gamma}$$

As

$$\int_{\mathcal{G}_{L/2}} \left| \frac{r}{\gamma} \right|^2 \leq C \int_0^{L/2} \frac{r^3 \mathrm{d}r}{\gamma(r)} \leq K[C_{ell}],$$

we get that, for a constant K depending on C_3^{reg} and C_{ell}, C_{cvx}, there holds:

$$|I_{in}[\mathbf{w}]| \leq K \|\mathbf{u}^*; H^2(B(\mathbf{0}, L))\| \|\nabla \mathbf{w}; L^2(\mathcal{F})\|.$$

This ends the proof. $\square$

4 Proof of Theorem 5

In this last section, we exhibit a particular case where the above informations yield a sharp asymptotic expansion of the quantity:

$$\int_{\mathcal{F}} |\nabla \mathbf{u}|^2.$$

Throughout this last section, we assume that $\Omega = \mathbb{R}^3 \setminus \mathcal{B}^*$ with $\mathcal{B}^*$ a sphere of radius R and that $\mathcal{B}$ is a sphere of radius S. We recall that we have then:

$$\begin{aligned}
\gamma_t(x,y) &= \frac{x^2+y^2}{2S} + O((x^2+y^2)^2)\,, \\
\gamma_b(x,y) &= -\frac{x^2+y^2}{2R} + O((x^2+y^2)^2)\,, \\
\gamma(x,y) &= \frac{x^2+y^2}{2R_1} - \frac{(x^2+y^2)^2}{8R_3^3} + O((x^2+y^2)^3)\,,
\end{aligned}$$

where R_1 and R_3 satisfy (22). We fix also a smooth boundary data $\mathbf{u}^*$ and denote by $\mathbf{u}$ the exact solution to the Stokes problem with boundary condition $\mathbf{u}^*$ and $\mathbf{v}[\mathbf{u}^*]$ the approximation that is constructed in Section 3.1.

We introduce the notations of the previous section: indices $0, 1, as, R$ distinguish the components of $\mathbf{u}^*$ and $\mathbf{u}$ and $\mathbf{v}[\mathbf{u}^*]$. Hence, we have

$$\mathbf{v}[\mathbf{u}^*] = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_R,$$

and a similar decomposition for $\mathbf{u}$. We decompose

$$\|\mathbf{u} - \mathbf{v}[\mathbf{u}^*]; V\| \leq \|\mathbf{u}_0 - \mathbf{v}_0; V\| + \|\mathbf{u}_1 - \mathbf{v}_1; V\| + \|\mathbf{u}_R; V\| + \|\mathbf{v}_R; V\|$$

Applying Proposition 19 and Proposition 22, Proposition 23 in the cylindrical case we deduce:

$$\int_{\mathcal{F}} |\nabla(\mathbf{u} - \mathbf{v}[\mathbf{u}^*])|^2 = O(1)\left\{\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2 + \|\mathbf{u}^*; H^{\frac{1}{2}}(\partial\Omega)\|^2\right\},$$

where we keep the convention that landau notations $O(1)$ stand for quantities depending on (h, R, S) which remains bounded by a constant depending on R, S only for $h \in (0,1]$.

The weak formulation of the Stokes problem (remarking that $\mathbf{u} - \mathbf{v}[\mathbf{u}^*]$ vanishes on $\partial\mathcal{F}$) yields also that:

$$\int_{\mathcal{F}} \nabla\mathbf{u} : \nabla(\mathbf{u} - \mathbf{v}[\mathbf{u}^*]|) = 0\,,$$

Hence, we have:

$$\int_{\mathcal{F}} |\nabla\mathbf{u}|^2 = \int_{\mathcal{F}} |\nabla\mathbf{v}[\mathbf{u}^*]|^2 + O(1)\left\{\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2 + \|\mathbf{u}^*; H^{\frac{1}{2}}(\partial\Omega)\|^2\right\}.$$

To compute the first integral on the right-hand side of this last equality, we split:

$$\int_{\mathcal{F}} |\nabla\mathbf{v}[\mathbf{u}^*]|^2 = \sum_{i=0,1,R} E_i + 2\left(E_{01} + E_{0R} + E_{1R}\right)\,,$$

where for $(i,j) \in \{0,1,R\}$ we define:

$$E_i = \int_{\mathcal{F}} |\nabla\mathbf{v}_i|^2 \qquad E_{ij} = \int_{\mathcal{F}} \nabla\mathbf{v}_i : \nabla\mathbf{v}_j\,.$$

We complete the proof by studying the asymptotics of all these integrals. The first-order terms will yield by computing E_0 and E_1.

4.1 Study of positive terms

We recall that, by applying Proposition 19, we get at first that

$$\int_{\mathcal{F}} |\nabla\mathbf{v}_R|^2 = O(1)\left\{\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2 + \|\mathbf{u}^*; H^{\frac{1}{2}}(\partial\Omega)\|^2\right\}. \tag{89}$$

Concerning the other terms, we remark that, by construction $\mathbf{v}_i = \mathbf{v}_{i,in} + \mathbf{v}_{i,ext}$ so that:

$$\int_{\mathcal{F}} |\nabla \mathbf{v}_i|^2 = \int_{\mathcal{F}} |\nabla \mathbf{v}_{i,in}|^2 + 2\int_{\Omega^*} \nabla \mathbf{v}_{i,in} : \nabla \mathbf{v}_{i,ext} + \int_{\Omega^*} |\nabla \mathbf{v}_{i,ext}|^2,$$

where, reproducing the computations in the proof of Lemma 18 (see (80)), we obtain that, for $i = 0,1$:

$$\int_{\Omega^*} |\nabla \mathbf{v}_{i,ext}|^2 + \int_{\Omega^*} |\nabla \mathbf{v}_{i,in}|^2 = O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2.$$

Hence, we get that, for $i = 0,1$:

$$E_i = E_{i,in} + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2, \quad E_{i,in} = \int_{\mathcal{F}} |\nabla \mathbf{v}_{i,in}|^2.$$

4.1.1 Asymptotics of $E_{1,in}$

Let first consider $\mathbf{v}_1$. We drop index 1 in the sequel and we recall that $\mathbf{v}_{1,in}$, denoted here by $\mathbf{v}_{in}$, is constructed as follows:

$$\begin{aligned}
\mathbf{v}_{in,//}(x,y,z) &= \frac{1}{2}(z-(h+\gamma_t))(z-\gamma_b)\nabla_{x,y}\, p_{in} - \left(\frac{z-(h+\gamma_t)}{\gamma}\right)\chi_L \mathbf{u}^*_{//}(\mathbf{0})\,,\\
v_{in,\perp}(x,y,z) &= \frac{1}{2}\mathrm{div}_{x,y}\left[\int_z^{h+\gamma_t} (s-(h+\gamma_t))(s-\gamma_b)\nabla_{x,y}\, p_{in}\,\mathrm{d}s\right]\\
&\quad + \int_z^{h+\gamma_t} \mathrm{div}_{x,y}\left[\left(\frac{s-(h+\gamma_t)}{\gamma}\right)\chi_L \mathbf{u}^*_{//}(\mathbf{0})\mathrm{d}s\right]\,,
\end{aligned}$$

for $(x,y,z) \in \mathcal{G}_L$, where:

$$p_{in}(x,y) = \chi_L(x,y)q_{in}(x,y)\,, \quad \forall\,(x,y,z) \in \mathcal{G}_L\,.$$

with χ_L a suitable truncation function and q_{in} the unique solution to

$$\begin{aligned}
-\frac{1}{12}\mathrm{div}_{x,y}(\gamma^3\nabla_{x,y}q_{in}) &= (x\partial_x u^*_\perp(\mathbf{0}) + y\partial_y u^*_\perp(\mathbf{0}))\mathbf{e}_z - \mathrm{div}_{x,y}\left[\frac{(\gamma_t+\gamma_b)}{2}\mathbf{u}^*_{//}(\mathbf{0})\right]\,, && \text{on } B(\mathbf{0},L)\,,\\
q_{in} &= 0\,, && \text{on } \partial B(\mathbf{0},L)\,.
\end{aligned}$$

From Proposition 11, we obtain first that:

$$E_1 = \int_{\mathcal{G}_{L/2}} |\nabla \mathbf{v}_{in}|^2 + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2,$$

hence we may replace p_{in} by q_{in} in computations from now on. Then, it comes from the proof of Proposition 21 (see (86)) that:

$$\int_{\mathcal{G}_{L/2}} |\nabla_{x,y}\mathbf{v}_{in,//}|^2 + |\nabla v_{in,\perp}|^2 = O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2. \tag{90}$$

We obtain:

$$E_{1,in} = \int_{\mathcal{G}_{L/2}} |\partial_z \mathbf{v}_{in,//}|^2 + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2.$$

Explicit computations yield that, on $\mathcal{G}_{L/2}$, there holds:

$$\partial_z \mathbf{v}_{in,//} = \frac{(z-(h+\gamma_t)) + (z-\gamma_b)}{2}\nabla_{x,y}q_{in} - \frac{\mathbf{u}^*_{//}(\mathbf{0})}{\gamma}.$$

Consequently, we have that: $E_{1,in} = E^1_{1,in} + E^2_{1,in}$ where (note that the cross-term vanishes by integrating w.r.t. z-variable at first)

$$E^1_{1,in} = \frac{1}{4}\int_{\mathcal{G}_{L/2}} |(z-(h+\gamma_t)) + (z-\gamma_b)|^2 |\nabla_{x,y}\tilde{q}_{in}|^2 , \quad E^2_{1,in} = \int_{\mathcal{G}_{L/2}} \frac{|\mathbf{u}^*_{//}(\mathbf{0})|^2}{\gamma^2}.$$

We end up the proof by computing the asymptotics of $E^1_{1,in}$ and $E^2_{1,in}$.

Concerning $E^2_{1,in}$, we expand, for sufficiently small r_0 :

$$\begin{aligned} E^2_{1,in} &= 2\pi|\mathbf{u}^*_{//}(\mathbf{0})|^2 \int_0^{L/2} \frac{r\mathrm{d}r}{\gamma(r)} \\ &= 2\pi|\mathbf{u}^*_{//}(\mathbf{0})|^2 \left(\int_0^{r_0} \frac{r\mathrm{d}r}{(h+\frac{r^2}{2R_1})} + O(1)\right) \\ &= 2\pi R_1|\mathbf{u}^*_{//}(\mathbf{0})|^2|\ln(h)| + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2 . \end{aligned}$$

As for $E^1_{1,in}$, we go back to the computations of Section 2. Indeed, integrating at first with respect to z, we get:

$$E^1_{1,in} = \frac{1}{12}\int_{B(\mathbf{0},L/2)} \gamma^3|\nabla q_{in}|^2,$$

where we apply Proposition 16 to q_{in} to compute the asymptotics of this last quantity. This yields:

$$E^1_{1,in} = \frac{24\pi}{5}R_1|\ln(h)||R_1\nabla_{x,y}u^*_{\perp}(\mathbf{0}) + \frac{(R-S)}{2(R+S)}\mathbf{u}_{//}(\mathbf{0})|^2 + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2 .$$

Finally, we obtain:

$$\begin{aligned} E_{1,in} = \left(2\pi R_1|\mathbf{u}^*_{//}(\mathbf{0})|^2 + \frac{24\pi}{5}R_1|R_1\nabla_{x,y}u^*_{\perp}(\mathbf{0}) + \frac{(R-S)}{2(R+S)}\mathbf{u}^*_{//}(\mathbf{0})|^2\right)|\ln(h)| \\ + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2. \end{aligned}$$

4.1.2 Asymptotics of $E_{0,in}$

We focus now on $E_{0,in}$ and drop index 0 for simplicity. Let first recall that $\mathbf{v}_{in}$ is constructed as:

$$\begin{aligned} \mathbf{v}_{in,//}(x,y,z) &= \frac{1}{2}(z-(h+\gamma_t))(z-\gamma_b)\nabla_{x,y}\, p_{in} , \\ v_{in,\perp}(x,y,z) &= \frac{1}{2}\mathrm{div}_{x,y}\left[\int_z^{h+\gamma_t} (s-(h+\gamma_t))(s-\gamma_b)\nabla_{x,y}\, p_{in}\,\mathrm{d}s\right] \end{aligned}$$

for $(x,y,z) \in \mathcal{G}_L$, where:

$$p_{in}(x,y) = \chi_L(x,y)q_{in}(x,y) , \quad \forall\, (x,y,z) \in \mathcal{G}_L .$$

with χ_L a suitable truncation function and q_{in} the unique solution to

$$\begin{aligned} -\frac{1}{12}\mathrm{div}(\gamma^3\nabla q_{in}) &= u^*_{\perp}(\mathbf{0}) \quad \text{on } B(\mathbf{0},L) , \\ q_{in} &= 0 , \qquad\quad \text{on } \partial B(\mathbf{0},L) . \end{aligned}$$

We recall further that, in this radial case, we may compute q_{in} explicitly:

$$q_{in}(r) = u^*_{\perp}(\mathbf{0})\int_r^L \frac{6s}{\gamma^3(s)}\mathrm{d}s , \quad \forall\, r \in (0,L) .$$

so that

$$\partial_r q_{in}(r) = -u^*_\perp(\mathbf{0})\frac{6r}{\gamma^3(r)}\,, \quad \forall\, r \in (0,L)\,.$$

This entails that $\mathbf{v}_{in} = v_r\mathbf{e}_r + v_z\mathbf{e}_z$ with:

$$\left.\begin{array}{rcl} v_r(x,y,z) & = & \dfrac{\partial_r q_{in}}{2}\left((z-(h+\gamma_t(r)))(z-\gamma_b(r))\right)\,, \\ & = & -u^*_\perp(\mathbf{0})\dfrac{3r}{\gamma^3(r)}\left((z-(h+\gamma_t(r)))(z-\gamma_b(r))\right)\,, \end{array}\right. \quad \forall\,(x,y,z) \in \mathcal{G}_{L/2}\,.$$

We assume from now on that $u^*_\perp(\mathbf{0}) = 1$ for simplicity. We note that we have

$$|u^*_\perp(\mathbf{0})| \leq O(1)\|\mathbf{u}^*; H^2(B(\mathbf{0},L))\|\,,$$

so that all $O(1)$ terms in the following computations will turn into $O(1)\|\mathbf{u}^*; H^2(B(\mathbf{0},L))\|^2$ in the final result.

As in the computations for $E_{1,in}$, from Proposition 11, we obtain first that:

$$E_{0,in} = \int_{\mathcal{G}_{L/2}} |\nabla \mathbf{v}_{in}|^2 + O(1),$$

and we replace p_{in} by q_{in}. Also, we already computed in the proof of Proposition 23 that

$$\int_{\mathcal{F}} |\nabla_{x,y} v_{in,\perp}|^2 = O(1)\,. \tag{91}$$

Consequently, we have:

$$E_{0,in} = \int_{\mathcal{G}_{L/2}} \left(|\partial_r v_r|^2 + \left|\frac{v_r}{r}\right|^2 + |\partial_z v_r|^2 + |\partial_z v_z|^2 \right) + O(1)\,,$$

where, due to the incompressibility condition satisfied by $\mathbf{v}_{in}$:

$$\begin{aligned} \int_{\mathcal{G}_{L/2}} |\partial_z v_z|^2 &= \int_{\mathcal{G}_{L/2}} \left|\frac{1}{r}\partial_r[r v_r]\right|^2 = \int_{\mathcal{G}_{L/2}} \left(|\partial_r v_r|^2 + \left|\frac{v_r}{r}\right|^2 \right) + \int_{r=L}\int_{z=\gamma_b(L)}^{z=h+\gamma_t(L)} |v_r|^2 \\ &= \int_{\mathcal{G}_{L/2}} \left(|\partial_r v_r|^2 + \left|\frac{v_r}{r}\right|^2 \right) + O(1) \end{aligned}$$

as v_r remains bounded independently of h away from the origin. We get thus:

$$E_{0,in} = I_z + 2I_r + O(1)\,,$$

with:

$$I_z = \int_{\mathcal{G}_{L/2}} |\partial_z v_r|^2\,, \qquad I_r = \int_{\mathcal{G}_{L/2}} \left(|\partial_r v_r|^2 + \left|\frac{v_r}{r}\right|^2 \right)\,.$$

Computation of I_z Replacing the integrand in I_z with its values yields:

$$\begin{aligned} I_z &= \frac{1}{4}\int_{B(\mathbf{0},L/2)}\int_{\gamma_b(r)}^{h+\gamma_t(r)} |\partial_r q_{in}(r)|^2 \left[(z-(h+\gamma_t(r))) + (z-\gamma_b(r))\right]^2 \mathrm{d}z\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{4}\left[\int_{B(\mathbf{0},L/2)} |\nabla q_{in}|^2\gamma^3 \mathrm{d}x\mathrm{d}y\right]\int_0^1 (2s-1)^2\mathrm{d}s \\ &= \frac{1}{12}\left[\int_{B(\mathbf{0},L/2)} |\nabla q_{in}|^2\gamma^3 \mathrm{d}x\mathrm{d}y\right]\,. \end{aligned}$$

At this point, we apply Proposition 15 to q_{in} yielding:

$$\left[\int_{B(\mathbf{0},L/2)} |\nabla q_{in}|^2 \gamma^3 \mathrm{d}x\mathrm{d}y\right] = 72\pi \left[\frac{R_1^2}{h} - 3\frac{R_1^4}{R_3^3}|\ln(h)|\right] + O(1).$$

This entails finally:

$$I_z = \frac{6\pi R_1^2}{h} - 18\pi\frac{R_1^4}{R_3^3}|\ln(h)| + O(1)\,.$$

Computation of I_r We proceed with the computation of $I_r = I_r^1 + I_r^2$ where:

$$I_r^1 := \int_{\mathcal{G}_{L/2}} |\partial_r v_r|^2\,, \qquad I_r^2 := \int_{\mathcal{G}_{L/2}} \left|\frac{v_r}{r}\right|^2\,.$$

We first compute I_r^2 by replacing v_r with its values:

$$\begin{aligned}
I_r^2 &= 18\pi \int_0^{L/2} \int_{\gamma_b(r)}^{h+\gamma_t(r)} \frac{[(z-(h+\gamma_t(r)))(z-\gamma_b(r))]^2}{\gamma(r)^6} r\mathrm{d}r\mathrm{d}z \\
&= 18\pi \int_0^{L/2} \frac{r\mathrm{d}r}{\gamma(r)} \int_0^1 [(s-1)s]^2 \mathrm{d}s \\
&= \frac{3\pi}{5} \int_0^{L/2} \frac{r\mathrm{d}r}{\gamma(r)}\,.
\end{aligned}$$

We already computed (see the computation of $E_{1,in}^2$) that:

$$\int_0^{L/2} \frac{r\mathrm{d}r}{\gamma(r)} = R_1|\ln(h)| + O(1)\,.$$

so that we obtain finally:

$$I_r^2 = \frac{3\pi}{5} R_1|\ln(h)| + O(1)\,.$$

Second, we expand $\partial_r v_r$. We have:

$$\partial_r v_r = \frac{1}{2}\left(\partial_{rr} q_{in}(r)(z-(h+\gamma_t(r)))(z-\gamma_b(r)) + \partial_r q_{in}(r)\partial_r[(z-(h+\gamma_t(r)))(z-\gamma_b(r))]\right).$$

Consequently:

$$\begin{aligned}
I_r^1 &= \frac{1}{4}\Big[\int_{\mathcal{G}_{L/2}} |\partial_{rr} q_{in}(r)(z-(h+\gamma_t(r)))(z-\gamma_b(r))|^2 \\
&\quad + \int_{\mathcal{G}_{L/2}} |\partial_r q_{in}(r)\partial_r[(z-(h+\gamma_t(r)))(z-\gamma_b(r))]|^2 \\
&\quad + 2\int_{\mathcal{G}_{L/2}} \partial_{rr} q_{in}\, \partial_r q_{in}\, \partial_r[(z-(h+\gamma_t(r)))(z-\gamma_b(r))](z-(h+\gamma_t(r)))(z-\gamma_b(r))\Big] \\
&= I_{r,a} + I_{r,b} + I_{r,c}
\end{aligned}$$

After tedious but straightforward computations, we get:

$$\begin{aligned}
I_{r,a} &= 15\pi R_1|\ln(h)| + O(1)\,, \\
I_{r,b} &= \left(24\pi R_1 - \frac{24\pi R_1^3}{RS}\right)|\ln(h)| + O(1)\,, \\
I_{r,c} &= -30\pi R_1|\ln(h)| + O(1)\,.
\end{aligned}$$

and

$$I_r = \frac{24\pi}{5}\left[2R_1 - \frac{5R_1^3}{RS}\right]|\ln(h)| + O(1)\,.$$

Combining computations of I_r and I_z, we obtain:

$$E_{1,in} = 6\pi\left[\frac{R_1^2}{h} + |\ln(h)|\left(\frac{16}{5}R_1 - \frac{8R_1^3}{RS} - 3\frac{R_1^4}{R_3^3}\right)\right] + O(1)\,.$$

4.2 Asymptotics of cross terms

We proceed with the computation of the asymptotics of cross terms E_{ij}, $i \neq j \in \{0,1,R\}^2$. We recall that with similar arguments as previously, there holds:

$$E_{ij} = E_{ij,in} + O(1)\left\{\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2 + \|\mathbf{u}^*; H^{\frac{1}{2}}(\partial\Omega)\|^2\right\},$$

with obvious notations.

4.2.1 Asymptotics of $E_{01,in}$

We first treat the term E_{01}. For this term, we assume without restricting the generality that $R\nabla u^*_\perp(\mathbf{0}) + \mathbf{u}^*_{//}(\mathbf{0})$ is parallel to $\mathbf{e}_1$. As a consequence, we obtain that $q_{1,in}$ reads $\varphi(r)\cos(\theta)$ in polar coordinates so that, as a function of (x,y) it satisfies:

$$q_{1,in}(-x,-y) = -q_{1,in}(x,y) \quad \forall\,(x,y) \in B(\mathbf{0},L).$$

Hence, introducing the rotation matrix,

$$S_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

there holds:

$$S_1\mathbf{v}_{1,in}(S_1(x,y,z)) = -\mathbf{v}_{1,in}(x,y,z)\,, \quad \forall\,(x,y,z) \in \mathcal{G}_L$$

On the opposite, we have that $q_{0,in}$ satisfies $q_{0,in}(-x,-y) = q_{0,in}(x,y)$ so that we have the symmetries:

$$S_1\mathbf{v}_{0,in}(S_1(x,y,z)) = \mathbf{v}_{0,in}(x,y,z)\,, \quad \forall\,(x,y,z) \in \mathcal{G}_L.$$

Going to the derivatives, this entails that:

$$E_{10,in} = \int_{\mathcal{G}_L} \nabla\mathbf{v}_{0,in} : \nabla\mathbf{v}_{1,in} = -\int_{\mathcal{G}_L} \nabla\mathbf{v}_{0,in} : \nabla\mathbf{v}_{1,in} = 0.$$

4.2.2 Asymptotics of $E_{0R,in}$

By definition, we have :

$$E_{0R,in} = \int_{\mathcal{G}_L} \nabla\mathbf{v}_{0,in} : \nabla\mathbf{v}_{R,in}$$

We introduce:

$$\tilde{p}_{R,in} = p_{R,in} - \int_{B(\mathbf{0},L)} p_{R,in}(x,y) v_{0,\perp,in}(x,y,\gamma_b(x,y))\mathrm{d}x\mathrm{d}y,$$

and, applying that $\mathbf{v}_{R,in}$ is divergence free, we transform:

$$\begin{aligned} E_{0R,in} &= \int_{\mathcal{G}_L} (\nabla\mathbf{v}_{R,in} - \tilde{p}_{R,in}\mathbb{I}_3) : \nabla\mathbf{v}_{0,in} \\ &= \int_{\mathcal{G}_L} (\nabla_{x,y}\mathbf{v}_{R,in,//} - \tilde{p}_{R,in}\mathbb{I}_2) : \nabla_{x,y}\mathbf{v}_{0,in,//} + \int_{\mathcal{G}_L} \partial_z\mathbf{v}_{R,in,//} \cdot \partial_z\mathbf{v}_{0,in,//} \\ &\quad - \int_{\mathcal{G}_L} \tilde{p}_{R,in}\partial_z v_{0,in,\perp} + \int_{\mathcal{G}_L} \partial_z v_{R,in,\perp} \cdot \partial_z v_{0,in,\perp} + \int_{\mathcal{G}_L} \nabla_{x,y}\mathbf{v}_{R,in,\perp} \cdot \nabla_{x,y}\mathbf{v}_{0,in,\perp} \end{aligned}$$

In this identity, we integrate by parts the integrals on the first line of the right-hand side yielding:

$$
\begin{aligned}
&\int_{\mathcal{G}_L} (\nabla_{x,y}\mathbf{v}_{R,in,//} - \tilde{p}_{R,in}\mathbb{I}_2) : \nabla_{x,y}\mathbf{v}_{0,in,//} + \int_{\mathcal{G}_L} \partial_z \mathbf{v}_{R,in,//} \cdot \partial_z \mathbf{v}_{0,in,//} \\
&= -\left(\int_{\mathcal{G}_L} (\Delta_{x,y}\mathbf{v}_{R,in,//} - \nabla_{x,y}\tilde{p}_{R,in}) \cdot \mathbf{v}_{0,in,//} + \int_{\mathcal{G}_L} \partial_{zz}\mathbf{v}_{R,in,//} \cdot \mathbf{v}_{0,in,//} \right) \\
&= -\left(\int_{\mathcal{G}_L} (\Delta_{x,y}\mathbf{v}_{R,in,//} - \nabla_{x,y}p_{R,in}) \cdot \mathbf{v}_{0,in,//} + \int_{\mathcal{G}_L} \partial_{zz}\mathbf{v}_{R,in,//} \cdot \mathbf{v}_{0,in,//} \right) \\
&= -\left(\int_{\mathcal{G}_L} \Delta_{x,y}\mathbf{v}_{R,in,//} \cdot \mathbf{v}_{0,in,//} \right)
\end{aligned}
$$

where we used that

- $\mathbf{v}_{0,in,//}$ vanishes on $\partial\mathcal{G}_L$
- $\partial_{zz}\mathbf{v}_{R,in,//} - \nabla_{x,y}p_{R,in} = 0$ in $\mathcal{G}_L$

We also note that $\tilde{p}_{R,in}$ does not depend on z and apply boundary conditions for $v_{0,\perp,in}$ yielding:

$$
\begin{aligned}
\int_{\mathcal{G}_L} \tilde{p}_{R,in}\partial_z v_{0,in,\perp} &= \int_{B(\mathbf{0},L)} \tilde{p}_{R,in} v_{0,in,\perp}(x,y,\gamma_b(x,y)) \\
&= 0.
\end{aligned}
$$

because of our choice for $\tilde{p}_{R,in}$. Finally, we have:

$$
\begin{aligned}
E_{0R,in} &= \int_{\mathcal{G}_L} \Delta_{x,y}\mathbf{v}_{R,in,//} \cdot \mathbf{v}_{0,in,//} + \int_{\mathcal{G}_L} \partial_z v_{R,in,\perp} \cdot \partial_z v_{0,in,\perp} \\
&\quad + \int_{\mathcal{G}_L} \nabla_{x,y} v_{R,in,\perp} \cdot \nabla_{x,y} v_{0,in,\perp}
\end{aligned}
$$

From (89) and (91), we have that:

$$
\int_{\mathcal{F}} |\nabla_{x,y} v_{0,in,\perp}|^2 + \int_{\mathcal{F}} |\nabla \mathbf{v}_{in,R}|^2 = O(1)(\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2).
$$

Consequently, it remains:

$$
\begin{aligned}
E_{0R,in} &= \int_{\mathcal{G}_L} \Delta_{x,y}\mathbf{v}_{R,in,//} \cdot \mathbf{v}_{0,in,//} + \int_{\mathcal{G}_L} \partial_z v_{R,in,\perp} \cdot \partial_z v_{0,in,\perp} \\
&\quad + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2
\end{aligned}
$$

Applying that $\mathbf{v}_{R,in}$ and $\mathbf{v}_{0,in}$ are incompressible, we get:

$$
\int_{\mathcal{G}_L} \partial_z v_{R,in,\perp} \partial_z v_{0,in,\perp} = -\int_{\mathcal{F}} \mathrm{div}_{x,y}\mathbf{v}_{0,in,//}\, \mathrm{div}_{x,y}\mathbf{v}_{R,in,//}
$$

As $\mathbf{v}_{0,//,in}$ vanishes on $\partial\mathcal{G}_L$, we integrate this identity by parts: leading to:

$$
\begin{aligned}
&\int_{\mathcal{G}_L} \Delta_{x,y}\mathbf{v}_{R,in,//} \cdot \mathbf{v}_{0,in,//} + \int_{\mathcal{G}_L} \partial_z v_{R,in,\perp} \cdot \partial_z v_{0,in,\perp} \\
&= \int_{\mathcal{G}_L} (\Delta_{x,y}\mathbf{v}_{R,in,//} + \nabla_{x,y}\mathrm{div}_{x,y}\mathbf{v}_{R,in,//}) \cdot \mathbf{v}_{0,in,//}
\end{aligned}
$$

so that forgetting remainder terms for conciseness:

$$
|E_{0R,in}| \leq \int_{\mathcal{F}} |\nabla^2_{x,y}\mathbf{v}_{R,in,//}||\mathbf{v}_{0,in,//}|.
$$

At this point, we recall the explicit form of $\mathbf{v}_{R,//,in}$:

$$\mathbf{v}_{R,in,//} = (z-(h+\gamma_t))(z-\gamma_b)\nabla p_{R,in} + \left(\frac{h+\gamma_t - z}{\gamma}\right)\chi_L \mathbf{u}^*_{R,//}.$$

Consequently, applying (33) there holds:

$$|\nabla^2_{x,y}\mathbf{v}_{R,in,//}| \leq K[R,S]\left(\gamma|\nabla p_{R,in}| + \gamma^{\frac{3}{2}}|\nabla^2 p_{R,in}| + \gamma^2|\nabla^3 p_{R,in}| \right.$$
$$\left. + |\nabla^2\mathbf{u}^*_{R,//}| + \frac{|\nabla\mathbf{u}^*_{R,//}|}{\gamma^{\frac{1}{2}}} + \frac{|\mathbf{u}^*_{R,//}|}{\gamma}\right)$$

We recall we have also:

$$|\mathbf{v}_{0,in,//}| \leq |u^*_{\perp}(\mathbf{0})|\frac{r}{\gamma}.$$

Finally, applying that $r \leq K[R,S]\gamma^{\frac{1}{2}}$, we get:

$$E^1_{0R,in} \leq K[R,S]|u^*_{\perp}(\mathbf{0})| \int_0^L \left(\gamma^{\frac{3}{2}}|\nabla p_{R,in}| + \gamma^2|\nabla^2 p_{R,in}| + \gamma^{\frac{5}{2}}|\nabla^3 p_{R,in}|\right) r\mathrm{d}r$$
$$+ \int_0^L \left(|\nabla^2\mathbf{u}^*_{R,//}| + |\nabla\mathbf{u}^*_{R,//}| + \frac{|\mathbf{u}^*_{R,//}|}{\gamma^{\frac{1}{2}}}\right) r\mathrm{d}r.$$

Hence:

$$E^1_{0R,in} \leq K[R,S]|u^*_{\perp}(\mathbf{0})|\left\{\left(\int_0^L \left(\gamma^3|\nabla p_{R,in}|^2 + \gamma^4|\nabla^2 p_{R,in}|^2 + \gamma^5|\nabla^3 p_{R,in}|^2\right) r\mathrm{d}r\right)^{\frac{1}{2}}\right.$$
$$\left. + \|\mathbf{u}^*_{R,//}; H^2(B(\mathbf{0},L))\|\right\},$$

so that, introducing $q_{R,in} = q^{(1)}_{R,in} + hq^{(2)}_{R,in}$ and applying Propositions 9 and 12, we obtain as for (85):

$$E^1_{0R,in} = O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2.$$

4.2.3 Asymptotics of $E_{1R,in}$

The computations of $E_{1R,in}$ follows the line of the preceding section. We first remark that (89) and (90) imply:

$$\int_{\mathcal{F}} |\nabla_{x,y}\mathbf{v}_{1,in}|^2 + |\nabla v_{1,in,\perp}|^2 + \int_{\mathcal{F}} |\nabla\mathbf{v}_{R,in}|^2 = O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2$$

so that:

$$E_{1R,in} = \int_{\mathcal{F}} \partial_z\mathbf{v}_{1,in,//}\,\partial_z\mathbf{v}_{R,in,//} + O(1)\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2.$$

Explicit formulas yield that, for $i = 1, R$, we have:

$$\partial_z\mathbf{v}_{i,in,//} = \nabla p_{i,in}((z-(h+\gamma_t)) + (z-\gamma_b)) + \frac{\mathbf{u}^*_{i,//}}{\gamma}.$$

Hence, integrating at first w.r.t. z and deleting vanishing terms, we obtain:

$$\begin{aligned}\left|\int_{\mathcal{F}} \partial_z \mathbf{v}_{1,in,//}\, \partial_z \mathbf{v}_{R,in,//}\right| &\leq K[R,S]\left(\int_{B(\mathbf{0},L)} |\nabla p_{R,in}||\nabla p_{1,in}|\gamma^3 + \int_{B(\mathbf{0},L)} \frac{|\mathbf{u}^*_{R,//}||\mathbf{u}^*_{1,//}|}{\gamma}\right) \\ &\leq K[R,S]\Big(\int_{B(\mathbf{0},L)} |\nabla p_{1,in}|^2\gamma^{\frac{7}{2}} + |\nabla p_{R,in}|^2\gamma^{\frac{5}{2}} \\ &\qquad + \int_{B(\mathbf{0},L)} \frac{|\mathbf{u}^*_{R,//}|^2}{\gamma^{\frac{3}{2}}} + \frac{|\mathbf{u}^*_{1,//}|^2}{\gamma^{\frac{1}{2}}}\,.\end{aligned}$$

As we already computed several times, the vanishing properties of $\mathbf{u}*_R$ in the origin imply that

$$\int_{B(\mathbf{0},L)} \frac{|\mathbf{u}^*_{R,//}|^2}{\gamma^{\frac{3}{2}}} + \frac{|\mathbf{u}^*_{1,//}|^2}{\gamma^{\frac{1}{2}}} \leq O(1)\|\mathbf{u}; H^2(B(\mathbf{0},L))\|^2$$

and, applying Proposition 9 and 10, we get:

$$\int_{B(\mathbf{0},L)} |\nabla q_{1,in}|^2\gamma^{\frac{7}{2}} + |\nabla q_{R,in}|^2\gamma^{\frac{5}{2}} \leq K[R,S]\|\mathbf{u}^*; H^3(B(\mathbf{0},L))\|^2$$

Hence, we have finally: $E_{1R,in} = O(1)\|\mathbf{u}; H^3(B(\mathbf{0},L))\|^2$.

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A Proof of Lemma 17

This appendix is devoted to the proof of

Lemma 24. *Given $R > 0$, there exists a $q \in C^\infty((0,\infty)) \cap C([0,\infty))$ solution to*

$$\partial_{ss} q + \left(\frac{1}{s} + \frac{3s}{R(1+\frac{s^2}{2R})} \right) \partial_s q - \frac{q}{s^2} = -\frac{12s}{(1+\frac{s^2}{2R})^3}\,, \qquad s \in (0,\infty)\,, \tag{92}$$

$$q(0) = 0 \quad \lim_{s\to\infty} q(s) = 0\,. \qquad . \tag{93}$$

Furthermore we have the asymptotic description:

$$q(s) = \frac{48R^3}{5s^3} + O(\frac{1}{s^4}) \quad \partial_s q(s) = -\frac{144R^3}{5s^4} + O(\frac{1}{s^5})\,. \tag{94}$$

Proof. From now on, we let $R > 0$ as in the statement of our lemma. As the proof seems standard, we only sketch the main steps.

First, we fix $L > 0$ and solve:

$$-\frac{1}{12}\left(\partial_{ss} q + \left(\frac{1}{s} + \frac{3s}{R(1+\frac{s^2}{2R})} \right) \partial_s q - \frac{q}{s^2} \right) = \frac{s}{(1+\frac{s^2}{2R})^3}\,, \qquad s \in (0,L)\,, \tag{95}$$

$$q(0) = 0, \quad q(L) = 0\,. \tag{96}$$

To this end, we introduce $\gamma_1(x,y) := (1+(x^2+y^2)/(2R))$ and introduce the bilinear form:

$$((\varpi,\varphi))_1 = \frac{1}{12}\int_{B(\mathbf{0},L)} |\gamma_1|^3 \nabla\varpi : \nabla\varphi$$

on $H^1_0(B(\mathbf{0},L))$. As γ_1 is smooth and does not vanish on $B(\mathbf{0},L)$ we obtain, by applying the Stampacchia theorem, existence of a unique $\varpi_L \in H^1_0(B(\mathbf{0},L))$ solution to

$$((\varpi_L,\varphi))_1 = \int_{B(\mathbf{0},L)} x\varphi(x,y)\mathrm{d}x\mathrm{d}y\,, \quad \forall\,\varphi \in H^1_0(B(\mathbf{0},L))\,.$$

Due to the invariance by rotation of γ_1 and the computation domain $B(\mathbf{0},L)$ for this weak formulation, we have that the unique solution reads in polar coordinates:

$$\varpi_L(s,\theta) = q_L(s)\cos(\theta) \quad \forall\,(s,\theta)\in(0,L)\times(-\pi,\pi)\,,$$

with $q_L \in C^\infty((0,L))\cap C([0,L])$ solution to (95)-(96).

Second, we remark that we have a maximum principle for (95) and that

$$\frac{s}{(1+\frac{s^2}{2R})^3} \leq \frac{(2R)^{\frac{3}{2}}}{s^2} \quad \forall s>0\,,$$

this yields in particular that

$$0 \leq q_L(s) \leq 12(2R)^{\frac{3}{2}}\,, \quad \forall\,s\in(0,L)\,, \quad \forall\,L>0\,.$$

Furthermore, setting $\bar{q}(s) = 48R^3/(5s^3)$ there holds:

$$-\frac{1}{12}\left(\partial_{ss}\bar{q} + \left(\frac{1}{s}+\frac{3s}{R(1+\frac{s^2}{2R})}\right)\partial_s\bar{q} - \frac{\bar{q}}{s^2}\right) = \frac{s}{(1+\frac{s^2}{2R})^3} + \varepsilon(s)$$

where

$$|\varepsilon(s)| \leq \frac{C_0}{s^6} \quad \forall\,s\in(1,\infty),$$

Setting also $\hat{q}(s) = 1/s^4$, there exists a constant $c>0$ such that:

$$-\frac{1}{12}\left(\partial_{ss}\hat{q} + \left(\frac{1}{s}+\frac{3s}{R(1+\frac{s^2}{2R})}\right)\partial_s\hat{q} - \frac{\hat{q}}{s^2}\right) \geq \frac{c}{s^6}. \quad \forall\,s>>1.$$

Finally, by application of the maximum principle, we obtain that there exists a constant K independent of L for which:

$$q_L(s) = \frac{48R^3}{5s^3} + q_{rem}(s) \quad \text{ with } |q_{rem}(s)| \leq \frac{K}{s^4}\,.$$

As the previous estimates are independent of L we can pass to the limit in $L\to\infty$ and obtain in the limit a solution to (92)-(93) with the expected zero-order asymptotic expansion. We do not detail further this passage to the limit. As γ_1 remains strictly positive on $(0,\infty)$ we may apply classical ellipticity results to yield that the solution-limit is indeed smooth on $(0,\infty)$ and continuous in 0.

Finally, we remark that $q' = \partial_s q$ satisfies:

$$\frac{1}{s}\partial_s(s(1+s^2/(2R))^3q'(s)) = \frac{q(s)(1+s^2/(2R))^3}{s^2} - 12s.$$

We obtain the expected asymptotic behavior of q' by integrating this equation between $s_1 = 1$ and $s_2 = s > 1$. □